

A
GEOMETRICAL TREATISE
OF
CONIC SECTIONS.
IN FOUR BOOKS.

TO WHICH IS ADDED,
A TREATISE ON THE PRIMARY PROPERTIES
OF
CONCHOIDS, THE CISSOID, THE QUADRATRIX,
CYCLOIDS, THE LOGARITHMIC CURVE,
AND THE
LOGARITHMIC, ARCHIMEDEAN, AND HYPERBOLIC SPIRALS.

BY THE
REV. ABRAM ROBERTSON, A.M. F.R.S.
SAVILIAN PROFESSOR OF GEOMETRY IN THE
UNIVERSITY OF OXFORD.

OXFORD:
AT THE CLARENDON PRESS.

1802.



10 AUG 1956

B622

A

48630



TO THE REVEREND

CYRIL JACKSON, D. D. F. R. S.

DEAN OF CHRIST CHURCH,

EQUALLY EMINENT FOR HIS OWN ABILITIES AND LEARNING,

AND FOR

HIS UNIFORM ENCOURAGEMENT AND PROMOTION

OF TALENTS AND ACQUIREMENTS IN OTHERS,

AS A

TESTIMONY OF THE HIGHEST ESTEEM FOR HIS CHARACTER,

AND AS A

TRIBUTE OF GRATITUDE FOR MANY IMPORTANT FAVOURS,

THIS WORK

IS MOST RESPECTFULLY INSCRIBED,

BY HIS MUCH OBLIGED AND FAITHFUL SERVANT,

THE AUTHOR.

ADVERTISEMENT.

THERE are two points of which it seems necessary that the reader of the following Treatise of Conic Sections should be apprized; first, what previous knowledge will be expected from him; and secondly, what extent of information the Treatise itself is intended to afford. The first of these will prevent the young student from entering upon the work till he is duly prepared, and the second will enable him to judge how far it is likely to contribute to the attainments which he has in view.

It is expected then, that the young student should understand thoroughly the first six Books of Euclid, the first twenty-one Propositions of the eleventh Book, the two first of the twelfth, and the first principles of Algebra, and Plane Trigonometry.

As no number can be assigned as a limit to the properties of the conic sections, any treatise on the subject can be supposed to contain only a selection of those which are most important and most useful, either generally, or with reference to the particular design of the Writer. In the present instance the design has been, to furnish the young Mathematician with

such a series of propositions as might prepare him for considering some of the most important truths in science, and enable him to enter on the study of natural philosophy, with the prospect of obtaining a thorough knowledge of the subject. According to these views the selection of properties and the extent of the work have been regulated; and at the same time the arrangement and division of the whole have been made with a design of accommodating two descriptions of readers. Those who are considered as constituting the first class are supposed to be desirous of a general but respectable portion of knowledge of the subject. For the use of such a perusal of the first three Books will be found sufficient, as they contain the properties of the sections most frequently referred to in pure and mixed mathematics. For those who rank under the second, or higher description, a knowledge of all the four Books will be requisite, as they complete the original design of rendering the whole a preparative for the Newtonian Philosophy. The Author flatters himself indeed, that he shall be found to have carried his elucidations of the Principia, in the present work, considerably beyond what have been attempted in other treatises of conic sections.

Something must now be added concerning the particular method, which has been adopted in these sheets, of deducing the primary properties of the sections from the nature of the cone.

*It is well known, that about the middle of the seventeenth century a difference of opinion took place among mathematicians concerning the proper source from which the properties of the conic sections should be deduced. But notwithstanding the objections which then began to be made to their deduction from the cone, and which have since been continued, it appears to the Author of this work that the difficulties attributed to the deductions from it were not to be imputed to the solid itself, but that they were occasioned solely by the manner in which the deductions had been made *. The early writers did not happen to perceive that the general and extensive property, expressed in the thirteenth Proposition of the first Book of this Treatise, could easily be obtained from the cone; and, not adverting to this, their deductions from the cone were sometimes tedious and intricate.*

*The above-mentioned property, as far as secants are concerned, occurs (I believe for the first time) in a folio volume, of which a treatise of conic sections makes a part, entitled, *Euclides Adauctus et Methodicus*, &c. published by Guarinus in 1671. The property to the same extent is to be found in Jones's *Synopsis Palmariorum Matheseos*, published in 1706; but neither of these two authors considered the property as a fundamental one, nor do they seem*

* For foundations for systems, independent of the cone, see the Scholium in page 110, and the first seven articles in the Scholium at the end of the third Book.

ADVERTISEMENT.

to have been aware of the advantages it was capable of producing. Its extensive utility was first evinced in Hamilton's *Conic Sections*, published in Latin in 1758; and on the appearance of this work objections to the cone ought to have ceased.

This was my persuasion when I published my former treatise*, and every deliberation on the subject since has tended to strengthen my conviction of its justice for the following reasons. First, the whole trouble with the cone is reduced to a very few demonstrations, for which no farther knowledge of Euclid is necessary than what is requisite for Spherical Trigonometry. Secondly, by this method the general properties are obtained with most ease and elegance. Lastly, by deducing the properties from the cone the treatise is rendered more extensively useful. No work on conic sections, confined to their description on a plane, can be applied to elucidations in Perspective, Projections of the Sphere, the Doctrine of Eclipses, and in some other particulars of the highest importance in science.

For the rest it need only be said, that the manner, in which the properties of the sections are classed and arranged, appeared to the Author, on the whole,

* In the year 1792 the author of the present work published a quarto volume, entitled, *Sectionum Conicarum Libri septem. Accedit Tractatus de Sectionibus Conicis, et de Scripturis quæ earum doctrinam tradiderunt*. The last mentioned Tract contains a full historical account of the subject.

to be that which was best calculated to shew what properties are general, and what are appropriate to each of the sections.

The treatise following that on conic sections, in the present volume, contains only the most common properties of the curves specified in the title page. It is intended as a preparative for those who wish to investigate the higher properties by means of Fluxions. In the first section methods of finding two mean proportionals and trisectioning an angle, by means of conchoids, are inserted. In the third section a method of dividing an angle in any given proportion, by means of the quadratrix, is given; as is also the quadrature of the circle, by means of the same curve.

N. B. When a demonstration, in the following work, is effected by means of two ranks of magnitudes, which, taken two and two in the same or in a cross order, have the same ratio to one another, they are placed thus,*

$$A : B : C : D$$

$$E : F : G : H ;$$

A, B, C, D representing the first rank, and E, F, G, H the second. Previous to this arrangement of the magnitudes, their ratio to one another is established, and therefore it evidently appears in which of the two orders the magnitudes are proportional. In

* That is either ex æquali, or ex æquali in proportionibus perturbata.

$$\left. \begin{array}{l} \text{t. } A B^2 \\ \text{or} \\ \text{f. } A B^2 \end{array} \right\}, \text{ which means the square of } \\
\text{A B if a tangent, or the rectangle under its segments} \\
\text{if a secant.}$$

ERRATA.

Page 14. in the Corollary *read* can cut a scalene cone

— 128. line 4. from the bottom, *for* it is a second diameter *read* c s. is a second diameter

— 126. line 9. *read* Nicomedes

— 140. line 21. *read* Dinoftratus

— 247. line 7. *for* B D *read* E D

— 247. in the margin *read* Fig. 13.

— 258. line 16, &c. *read* analogous to a series of logarithms, and the ordinates B F, C G, D H, &c. are analogous to the natural numbers of these logarithms.

— 259. line 15. and 18. *for* curve *read* spiral

LEMMAS

LEMMA S

FOR THE

FIRST THREE FOLLOWING BOOKS

OF

CONIC SECTIONS.

LEMMA I.

If the plane figure $A F D$ be bounded by the straight line $A D$ and the curve $A F D$, and if the square of the straight line $F I$, drawn from any point F in the curve perpendicular to $A D$, be equal to the rectangle under the segments $A I$, $I D$, the figure will be a semicircle. FIG. 1.

For let $A D$ be bisected in c , and draw $c F$. Then (47. i.) the square of $c F$ is equal to the squares of $c I$, $I F$ together, and therefore, by hypothesis, equal to the square of $c I$ together with the rectangle under $A I$, $I D$. But as $A D$ is bisected in c , the square of $A c$ or $c D$ (5. ii.) is equal to the square of $c I$ together with the rectangle under $A I$, $I D$. Consequently the square of $c F$ is equal to the square of $A c$ or $c D$; and therefore $c F$ is equal to $A c$ or $c D$. The figure $A F D$ is therefore a semicircle. For Prop. IV. Book I.

If two straight lines be parallel, and a plane pass through each of them, the common section of these planes, if they cut one another, will be parallel to each of the parallel lines.

Fig. 2.

Let two straight lines $A D, B C$ be parallel. Let the plane $A D F$ pass through $A D$, and cut the plane $B C F$, passing through $B C$, in the straight line $E F$; the line of common section $E F$ is parallel to $A D, B C$.

For let these planes be cut by two parallel planes $E A B, F D C$. Let the plane $E A B$ cut the plane $A B C D$ in the straight line $A B$; the plane $A D F E$ in $A E$, and the plane $B C F E$ in $B E$. Let the plane $F D C$ cut the plane $A B C D$ in the straight line $D C$, the plane $A D F E$ in $D F$, and the plane $B C F E$ in $C F$. Then the straight line $A B$ is parallel to $D C$ (16. xi.) $A E$ is parallel to $D F$, and $B E$ is parallel to $C F$. Hence (34. i.) $A B, D C$ are equal; the angle $E A B$ (10. xi.) is equal to the angle $F D C$, and the angle $E B A$ is equal to the angle $F C D$. Consequently (26. i.) $A E, D F$ are equal, and therefore (33. i.) $A D, E F$ are equal and parallel. For the same reasons $E F, B C$ are parallel. For Prop. VII. Book I.

LEMMA III.

If a straight line cut either of two parallel straight lines, and be in the same plane with them, it will, if sufficiently produced, cut the other.

Fig. 3.

Let the straight lines $A B, D E$ be parallel, and let $C F$ be in the same plane with them, and cut $A B$ in the point c ; the straight line $c F$ produced will cut $D E$.

For let any point G be taken in $D E$, and draw $c G$. Then as the angles $B c G, E G c$ together (29. i.) are equal to two right angles, the angles $F c G, E G c$ together

gether are less than two right angles, and therefore (ax. 12. i.) the straight lines CF , DE produced will meet one another. For Prop. VIII. Book I.

LEMMA IV.

If two straight lines cutting one another be parallel to a plane, a plane passing through them will be parallel to the same plane.

Let the two straight lines AB , CB , cutting one another in B , be parallel to the plane $DGHE$; the plane passing through AB , CB is parallel to the plane $DGHE$.

Fig. 4

For let F be any point in the plane $DGHE$. Through AB and F let a plane be passed, and let it cut the plane $DGHE$ in the straight line DFH ; and let a plane passing through CB and F cut it in EFG . Then will AB be parallel to DFH , and CB will be parallel to EFG . For if not, then AB will meet DFH , and CB will meet EFG , and consequently AB , CB will meet the plane $DGHE$, in which DH , EG are, contrary to the hypothesis. The plane passing through AB , CB (15. xi.) is therefore parallel to the plane $DGHE$. For Prop. X. Book I.

LEMMA V.

If the first of eight straight lines be to the second as the third to the fourth, and if the fifth be to the sixth as the seventh to the eighth; then the rectangle under the first and fifth will be to the rectangle under the second and sixth as the rectangle under the third and seventh to the rectangle under the fourth and eighth. And if the rectangle under the first and fifth, of eight straight lines, be to the rectangle under the second and sixth as the rectangle under the third and seventh to the rectangle under the fourth and eighth, and if the first be to the second as

B 2

the

LEMMAS FOR THE

the third to the fourth, then the fifth will be to the sixth as the seventh to the eighth.

Fig. 5.

Part I. Let AB , the first of eight straight lines, be to BC the second, as DE the third to EF the fourth, and let GB the fifth be to BH the sixth as IE the seventh to EK the eighth; then the rectangle under AB , GB is to the rectangle under BC , BH as the rectangle under DE , IE to the rectangle under EF , EK .

For let AB , BC be in a straight line; GB , BH be in a straight line; DE , EF be in a straight line; and IE , EK be in a straight line; and let these straight lines be at right angles to one another, and let the rectangles be completed as represented in the figure. Then AG is the rectangle under AB , GB ; CH is the rectangle under BC , BH ; DI is the rectangle under DE , IE ; and FK is the rectangle under EF , EK . By hypothesis $AB : BC :: DE : EF$, and therefore (II. v. and I. vi.) $AG : GC :: DI : IF$. Again, by hypothesis, $GB : BH :: IE : EK$, and therefore (II. v. and I. vi.) $GC : CH :: IF : FK$. Consequently,

$$AG : GC : CH$$

$$DI : IF : FK,$$

and therefore (22. v.) $AG : CH :: DI : FK$.

Fig. 5.

Part II. The construction, with respect to the rectangles, remaining as stated above, let the rectangle AG be to the rectangle CH as the rectangle DI to the rectangle FK , and let AB be to BC as DE to EF ; then GB is to BH as IE to EK .

For, by hypothesis and inversion, $BC : AB :: EF : DE$; and therefore (II. v. and I. vi.) $GC : AG :: IF : DI$. Again by hypothesis $AG : CH :: DI : FK$. Consequently,

$$GC : AG : CH$$

$$IF : DI : FK;$$

and

and therefore (22. v.) $GC : CH :: IF : FK$. But (1. vi.) $GC : CH :: GB : BH$, and $IF : FK :: IE : EK$; and therefore (11. v.) $GB : BH :: IE : EK$. For Prop. X. Book I.

LEMMA VI.

If the points C, D be so situated in the straight line A B, that the rectangle D A C is equal to the rectangle C B D, then A C is equal to B D: or if the rectangle A C B be equal to the rectangle B D A, then A C is equal to B D. Fig. 6.

Case I. Let CD be bisected in E , and then (6. ii.) the rectangle $D A C$ together with the square of EC is equal to the square of AE ; and the rectangle $C B D$ together with the square of ED is equal to the square of BE . The squares of AE , BE are therefore equal, and consequently AE is equal to BE , and AC is equal to BD .

Case II. Let AB be bisected in E , and then (5. ii.) the rectangle $A C B$ together with the square of EC is equal to the square of AE or EB ; and the rectangle $B D A$ together with the square of ED is equal to the square of EB . The squares of EC , ED are therefore equal, and consequently EC is equal to ED , and AC is equal to BD . For Prop. I. Book II.

LEMMA VII.

If a straight line touch a circle, and two straight lines cutting the circle pass through the point of contact, and meet a straight line parallel to the tangent, the rectangle under the segments of the one, between the point of contact and circumference, and between the point of contact and straight line parallel to the tangent, will be equal to the rectangle under the segments of the other, between the point of contact and circumference and between the point of contact and straight line parallel to the tangent.

B 3

Let

Fig. 7.
and
8.

Let the straight line AB be parallel to the straight line RG touching the circle EPF in the point P , and let the two straight lines EP , FP , passing through P , meet the circumference again, the one in E and the other in F , and let EP meet the straight line AB in C , and FP meet it in D ; then the rectangle under EP , PC is equal to the rectangle under FP , PD .

For EF being drawn, the angle EPF (32. iii.) is equal to the angle RPE , which (29. i.) is equal to the angle PCD . The triangles EPF , PCD are therefore equiangular, and (4. vi.) $EP : PF :: DP : PC$. Consequently, (16. vi.) the rectangle under EP , PC is equal to the rectangle under FP , PD *. For Prop. I. Book III.

Cor. 1. If AB cut the circle in B , and PB be drawn, it may be proved in the same way that the rectangle under FP , PD is equal to the square of PB . For the tangent RP being produced to G , and BF being drawn, the angle BPG (32. iii.) is equal to the angle BFP ; and (29. i.) it is also equal to the angle DBP . The triangles BFP , DBP are therefore equiangular, and (4. vi.) $FP : PB :: PB : PD$, and (17. vi.) the rectangle under FP , PD is equal to the square of PB .

Fig. 8.

Cor. 2. The rest remaining as above, if BG be drawn parallel to FP , BG is (34. i.) equal to PD , and therefore by the above $FP = \frac{PB^2}{BG}$.

LEMMA VIII.

If the first of three straight lines be to the third as the square of the sum of the first and second to the square of the sum of the second and third, the second will be a

* The straight line AB may be on either side of the tangent RG , and it is not necessary, upon being produced indefinitely, that it should meet the circumference of the circle.

mean

first and second to the square of the difference of the second and third; the second will be a mean proportional between the first and third.

Part I. Let A denote the first, B the second, and C the third, of the straight lines. Then by hypothesis $A : C :: A + B^2 : B + C^2$; and it is to be proved that B is a mean proportional between A and C .

Let D be a mean proportional between A and C , and then, by inversion, $D : A :: C : D$, and (18. v.) $A + D : A :: D + C : D$; and therefore by alternation $A : D :: A + D : D + C$. Consequently (22. vi.) $A^2 : D^2 :: A + D^2 : D + C^2$. But (Cor. 2. 20. vi.) $A : C :: A^2 : D^2$, and therefore by hypothesis (and 11. v.) $A + B^2 : B + C^2 :: A + D^2 : D + C^2$. Consequently (22. vi.) $A + B : B + C :: A + D : D + C$; and therefore, by conversion, $A + B : A - C :: A + D : A - C$, and (14. v.) $A + B$ is equal to $A + D$. Consequently B is equal to D , and therefore B is a mean proportional between A and C .

Part II. Let A denote the first of the three straight lines, B the second, and C the third. Then by hypothesis $A : C :: A - B^2 : B - C^2$; and it is to be proved that B is a mean proportional between A and C .

Let D be a mean proportional between A and C . Then $A : D :: D : C$, and by conversion $A : A - D :: D : D - C$; and by alternation, $A : D :: A - D : D - C$. Consequently (22. vi.) $A^2 : D^2 :: A - D^2 : D - C^2$. But (Cor. 2. 20. vi.) $A : C :: A^2 : D^2$; and therefore by hypothesis (and 11. v.) $A - B^2 : B - C^2 :: A - D^2 : D - C^2$. Consequently (22. vi.) $A - B : B - C :: A - D : D - C$; and therefore (18. v.) $A - C : B - C :: A - C : D - C$. Hence (14. v.) $B - C$ is equal to

$B - C$

$D - C$,



$b = c$, and therefore b is equal to d . Consequently b is a mean proportional between a and c . For Prop. XXIII. Book III.

A
GEOMETRICAL TREATISE
OF
CONIC SECTIONS.

BOOK I.

Containing general Properties deduced from the Cone.

DEFINITIONS.

I.

IF through the point v , without the plane of the circle $A F B$, a straight line $A v D$ be drawn, and extended indefinitely both ways, and if the point v remain fixed, and the straight line $A v D$ be moved round the whole circumference of the circle, two Superficies will be generated by its motion, each of which is called a *Conical Superficies*; and these mentioned together are called *Opposite Superficies*. Fig. 9.

Cor. A straight line drawn from the fixed point v to any point G in either superficies is wholly in that superficies; and, being produced, the part on the other side of v is wholly in the opposite superficies. For a straight line

superficies; and, being produced, the part beyond v is in the opposite superficies. Hence the Cor. is evident; for only one straight line can be drawn from v to G , as two straight lines cannot inclose a space.

II.

The solid contained by the conical superficies and the circle $A F B$ is called a *Cone*.

III.

The fixed point v is called the *Vertex of the Cone*.

IV.

The circle $A F B$ is called the *Base of the Cone*.

V.

Any straight line drawn through the vertex of the cone to the circumference of the base is called a *Side of the Cone*.

VI.

A straight line $v c$, drawn through the vertex of the cone and the center of the base, is called the *Axis of the Cone*.

VII.

If the axis of the cone be perpendicular to the base, it is called a *Right Cone*.

VIII.

If the axis of the cone be not perpendicular to the base, it is called a *Scalene Cone*.

IX.

A plane is said to *touch a conical superficies*, when it meets the superficies, and when, being produced indefinitely, in any direction, it falls without the superficies.

X.

A straight line which meets a conical superficies, and which, being produced both ways, falls without the superficies.

superficies, is called a *Tangent*; but a straight line which meets a conical superficies in two points, or each of the opposite superficies in one point, is called a *Secant*. BOOK
I.

XI.

A straight line is said to be parallel to a plane, when both being produced ever so far, both ways, they do not meet,

XII.

If a cone be cut by a plane, their common intersection is called a *Conic Section*.

XIII.

The common intersection of any plane, not passing through the vertex of the cone, with the conical superficies, is called the *Curve of a Conic Section*.

PROP. I.

If a cone be cut by a plane passing through the vertex, the section will be a triangle.

Let the cone $VAFB$ be cut by a plane passing through V the vertex, and let VAB be the common intersection of the cone and plane; the section VAB is a triangle. Fig. 9.

For let the plane, passing through V , cut the plane of the base in the straight line (3. xi.) AB , and the circumference of the base in the points A, B ; and let the straight lines VA, VB be drawn. Then, as the points V, A, B are in the plane cutting the cone, the straight lines VA, VB are wholly in the same plane; and as the points A, B are in the conical superficies, the straight lines VA, VB are also wholly in the superficies, by the corollary to the first Definition. The straight lines VA, VB are therefore the common intersections of the conical superficies and the plane cutting the cone; and consequently the section VAB is a triangle.

Cor.

BOOK
I.

Cor. If a plane, passing through the vertex, cut a cone, it will cut the opposite superficies in two straight lines, and only in those two. For if the plane VAB be extended on both sides of the vertex, it will cut the opposite superficies in the straight lines VA , VB produced, and in them only. This is evident from the above, and the corollary to the first Definition.

PROP. II.

If either of the opposite conical superficies be cut by a plane parallel to the base of the cone, the common intersection of the superficies and the plane will be the circumference of a circle, and its center will be in the axis of the cone.

Fig. 10.

Let the superficies $VABD$, or its opposite superficies, be cut by a plane parallel to the base ABD of the cone, and let FGH be the common intersection of the superficies and this plane; FGH is the circumference of a circle, and its center is in VC , the axis of the cone, or in VC produced.

Let c be the center of the base, and let the axis VC cut the plane FGH in the point I . From the point I , and in the plane FGH , draw any two straight lines IF , IG to the conical superficies. Through VIC , IF let a plane be passed, and let it cut the superficies in the side VFA , and the base of the cone in the straight line CA . Let a plane also be passed through VIC , IG , and let it cut the superficies in the side VGB , and the base of the cone in the straight line CB . Then (16. xi.) FI , AC , and also GI , BC are parallel to one another, each to each: and (29. i.) the triangles ACV , FIV , and also the triangles BCV , GIV are equiangular, each to each. Consequently (4. vi.) $AC : FI :: VC : VI$, and also $VC : VI :: BC : GI$; and therefore (11. v.) $AC : FI :: BC : GI$, and as AC is equal to BC , FI is (14. v.) equal to GI . In the same manner it may be proved that

Fig. 1.

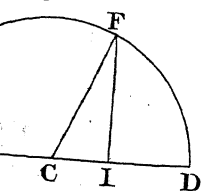


Fig. 2.

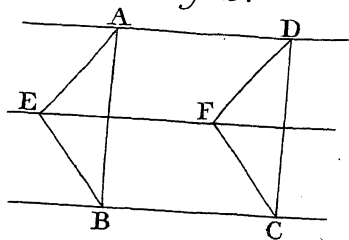


Fig. 3.

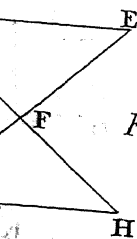
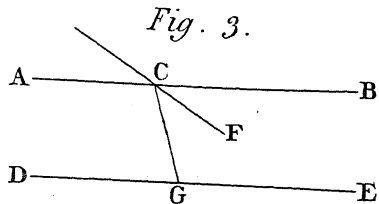


Fig. 4.

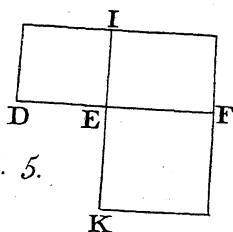
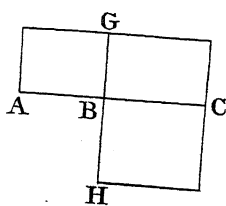
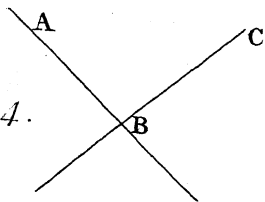


Fig. 5.

Fig. 6.

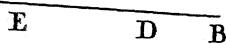


Fig. 7.

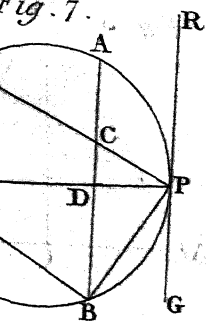


Fig. 8.

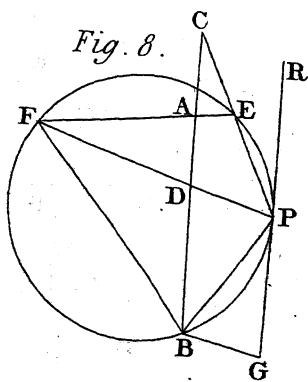
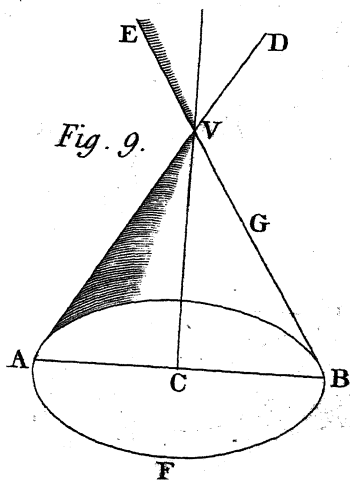
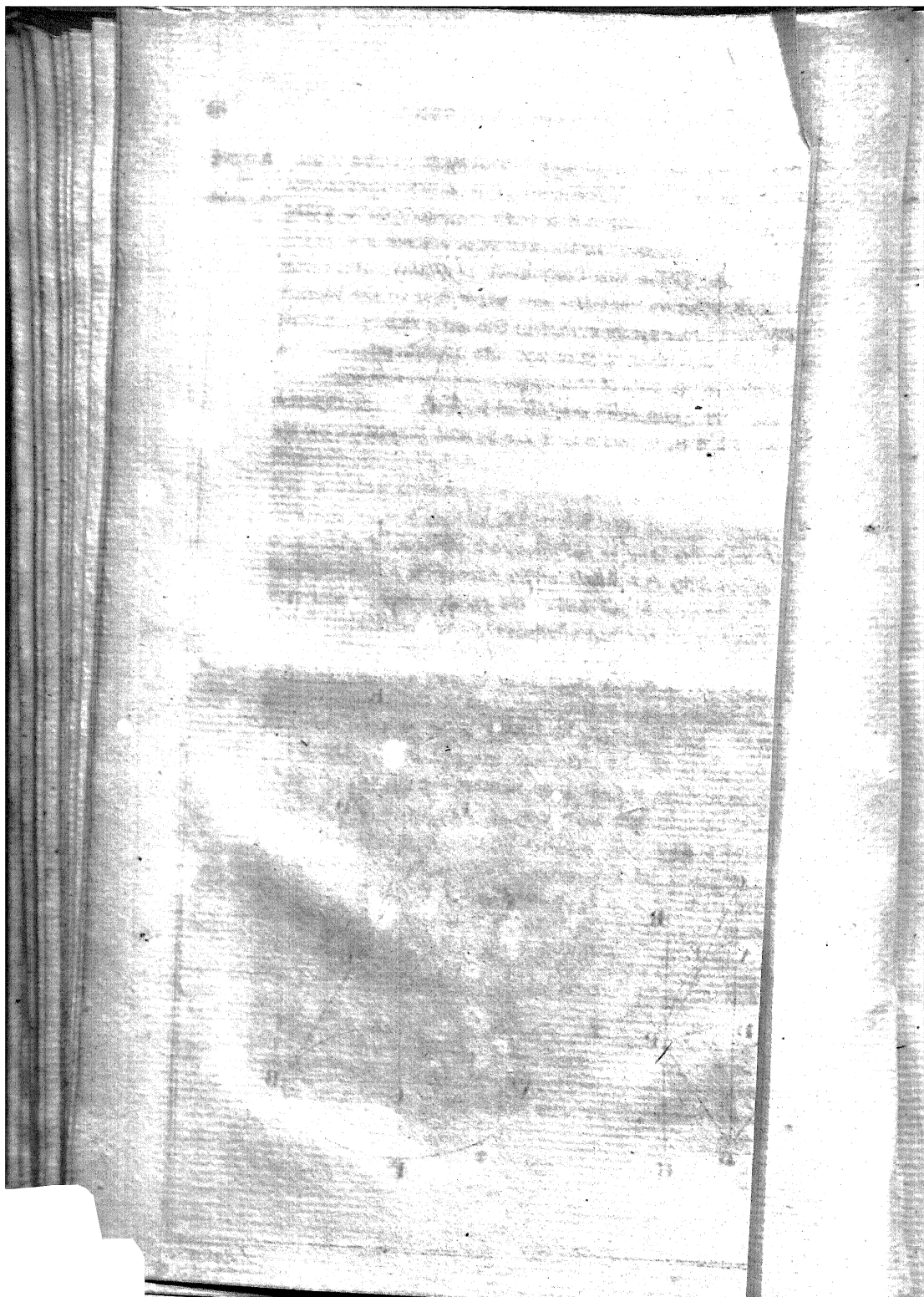


Fig. 9.





that any other straight line drawn in the plane $F G H$ from the point I to the conical superficies is equal to $F I$; and therefore $F G H$ is the circumference of a circle, and the point I , in the axis $V C$, is its center.

Cor. 1. From this Proposition, and the first Definition, it appears that any circle parallel to the base of the cone, having its center in the axis, and its circumference in either of the opposite superficies, may be taken for the base of the cone.

Cor. 2. The solid contained by the conical superficies $V F G H$, opposite to $V A B D$, and the circle $F G H$, is a cone.

PROP. III.

If a scalene cone be cut through the axis by a plane perpendicular to the base, of the sides of the section, meeting in the vertex, one will be the greatest, and the other the least of all the sides of the cone.

For let $V N O P$ be a scalene cone, of which V is the vertex, $N O P$ the base, and C the centre of the base. Let the straight line $V B$ be perpendicular to the plane of the base, and meet it in B . Draw the straight line $B C$, and let it meet the circumference of the base in the points P, N . Through the straight lines $V B, B C$ let a plane be passed, and let it cut the cone; and let the section formed, with the cone, be the triangle $V N P$, as in the first Proposition. Then as the plane of the triangle $V N P$ passes through C , it cuts the cone through the axis*; and, as it passes through $V B$, it is also (18. xi.) perpendicular to the

Fig. 11.
and
12.

* As the representation of the axis could not render the demonstration more perspicuous, it was intentionally omitted in the figures. The Reader will find the same omission in the figure for Prop. IV. and V. and intentionally made, for the same reason.

base

BOOK base NOP . If therefore the point B be farther from N than from P , it remains to be proved, that VN is greater and VP less than any other side of the cone.

I.

Let VO be any other side of the cone, and let it meet the circumference of the base in O , and draw BO . Then, as VB is perpendicular to the plane of the base, the angles VBN , VBP , VBO are right angles; and therefore (47. i.) the square of VN is equal to the squares of VB , BN together; and the square of VO is equal to the squares of VB , BO together; and the square of VP is equal to the squares of VB , BP together. But (7. and 8. iii.) BN is greater than BO , and BO is greater than BP ; and therefore the square of BN is greater than the square of BO , and the square of BO is greater than the square of BP . Consequently the square of VN is greater than the square of VO , and the square of VO is greater than the square of VP . Of all the sides of the cone therefore, VN is the greatest, and VP is the least.

If the perpendicular VB fall into the circumference of the base, then B and P will coincide; and (referring to 15. iii. instead of 7. and 8. iii) the demonstration will be the same as above*.

Cor. As there can be only one perpendicular to a plane (13. xi.) drawn from the same point above the plane, it is evident from the demonstration of this Proposition that only one plane can cut a cone through the axis, and be perpendicular to the base.

* It is evident that all the sides of a right cone are equal to one another. For, in this case, the perpendicular to the base, drawn from V , will fall into C , the centre, according to the seventh definition.

BOOK
I.

PROP. IV.

Let the scalene cone $VNOP$ be cut by a plane passing through the axis, and perpendicular to the base NOP , and let the common section be the triangle VNP ; in the side VP take any point D , and in the plane of the triangle make the angle VDA equal to the angle VNP ; then if the cone be cut by a plane passing through DA , and perpendicular to the triangle VNP , its common section $AEDB$ with the cone will be a circle.

Fig. 13.

For let the side VP be less than the side VN , as in the preceding Proposition, and produce AD to T and NP to R . Then, as VN is greater than VP , the angle VPN (18. i.) is greater than the angle VNP . But the angles VNP , VDA are equal, by hypothesis; and as the angle PDT is equal (15. i.) to the angle VDA , the angle VPN is greater than the angle PDT . The angles VPN , DPR together are therefore greater than the angles PDT , DPR together; and consequently the angles PDT , DPR together are less than two right angles. If therefore the straight lines AD , NP be sufficiently produced they will meet. Let them be produced and meet in R ; and let the plane of the section $AEDB$ cut the plane of the base NOP in the straight line RS . In DA take any point I , and let the cone be cut by a plane passing through I and parallel to the base; and let the section formed be the circle $HFKB$, as in the second Proposition. Let the circle $HFKB$ cut the triangle VNP in the straight line HIK , and the section $AEDB$ in the straight line FIB . Then as the section $HFKB$ is parallel to the plane of the base, and as these parallel planes are cut by the plane of the section $AEDB$, the common sections (16. xi.) FIB , SR are parallel; and as the plane of the section $AEDB$, and the plane of the base NOP are perpendicular to the plane

BOOK I. plane of the triangle VNP , and cut one another in SR , the straight line SR (19. xi.) is perpendicular to the plane of the triangle VNP . The straight line FIB (8. xi.) is therefore perpendicular to the plane VNP , and consequently (4. xi.) perpendicular to HK , AD . But, as the plane of the triangle VNP passes through the axis, HK is a diameter of the circle $HFKB$ by the second Proposition, and therefore (3. iii.) FB is bisected in I . Consequently (35. iii.) the rectangle under KI , IH is equal to the square of FI or BI . Again, as the circle $HFKB$ is parallel to the plane of the base, and as these parallel planes are cut by the triangle VNP , the common sections (16. xi.) HIK , NP are parallel. The angle AHI (29. i.) is therefore equal to the angle VNP , and consequently equal to the angle KDI . The angles (15. i.) AIH , KID are also equal, and therefore the triangles AIH , KID are equiangular. Consequently (4. vi.) $AI : IH :: KI : ID$, and (16. vi.) the rectangle under AI , ID is equal to the rectangle under KI , IH ; and therefore, by the above, the rectangle under AI , ID is equal to the square of FI or BI . The section $AFDB$ is therefore a circle, by the first Lemma.

The circle $AFDB$ formed in a scalene cone, in the manner mentioned in the Proposition, is called a *Subcontrary Section*.

PROP. V.

If a conic section be a circle, and be not parallel to the base of the cone, it will be a subcontrary section.

Fig 13.

Let the cone $VNOP$ be cut by a plane not parallel to the base NOP , and let the section $AFDB$ formed by it, with the cone, be a circle; $AFDB$ is a subcontrary section.

For let I be the point in which the axis of the cone meets

meets the circle $A F D B$, and through I let a plane be passed parallel to the base, and let $F K B H$ be the circle formed by it with the cone, as in the second Proposition. Let $B F$ be the common section of this circle with the circle $A F D B$. Then by Prop. II. the point I is the center of the circle $F K B H$, and consequently $B F$ is bisected in I . Through I draw in the circle $A F D B$ the straight line $A D$ at right angles to $B F$; and through $A D$ and V , the vertex, let a plane be passed, and let $V N P$ be the triangle formed by it with the cone, as in the first Proposition. Let $H K$ be the line of common section of the triangle $V N P$ and the circle $F K B H$. Let L be any point in $A D$, and through L let a plane be passed parallel to the base $N O P$, or to the circle $F K B H$. Let $M C E G$ be the circle formed by this plane with the cone, and let $M E$ be its line of common section with the triangle $V N P$, and $C L G$ its line of common section with the circle $A F D B$. Then (16. xi.) the straight lines $H K$, $M E$ are parallel, as are also $B I F$, $C L G$; and as $A I B$ is a right angle, $A L C$ is (29. i.) a right angle. Again, as the straight line $A D$ bisects the straight line $B F$ at right angles, $A D$ (Cor. I. iii.) is the diameter of the circle $A F D B$. The straight line $G C$ (3. iii.) is therefore bisected in L . But as I is the point in which the axis of the cone meets the circle $A F D B$, it is evident that the triangle $V N P$ cuts the cone through the axis, and consequently by Prop. II. $M E$ is a diameter of the circle $M C E G$, and the point L is not its center. Hence the diameter $M E$ (3. iii.) bisects $G C$ in L at right angles, and $G L$ is at right angles to $A D$, $M E$, and therefore it is at right angles to the triangle (4. xi.) $V N P$. Consequently (18. xi.) each of the sections $A F D B$, $M G E C$ is at right angles to the triangle $V N P$, and therefore as $M G E C$ is parallel to the base, the cone is cut by the plane

BOOK I. $\vee N P$ passing through the axis of the cone, and perpendicular to the base $N O P$. Again as $G L$ is at right angles to each of the two diameters $M E$, $A D$, the rectangle under $M L$, $L E$ is equal to the rectangle under $D L$, $L A$, each of these rectangles (35. iii.) being equal to the square of $G L$; and therefore (16. vi.) $D L : L E :: M L : L A$, and (6. vi.) the angle $L D E$ is equal to the angle $A M L$, or (29. i.) $\vee N P$. The circle $A P D E$ is therefore a subcontrary section.

Cor. A conic section neither parallel to the base of the cone, nor a subcontrary section, is not a circle.

DEFINITIONS.

XIV.

Fig. 10. The cones $\vee A E D$, $\vee F G H$, having the common vertex \vee , and whose superficies are opposite, being generated by the same line as in the first Definition, are called *Opposite Cones*.

Cor. It is evident from this, and the second Proposition, that if either of the opposite cones be cut by a plane parallel to the base of either, the section will be a circle.

XV.

Fig. 15. If the plane $\vee B E$ touch the conical superficies in the side $\vee B$, and the cone $\vee A B F$ be cut by the plane $F D C$ parallel to the plane $\vee B E$, the section $F D C$, formed by the cutting plane and the cone, is called a *Parabola*.

XVI.

The plane $\vee B E$ is called the *Vertical Plane to the Parabola*.

Cor. I. As the cone may be indefinitely extended, it is evident that the parabola may also be indefinitely extended; and as the parabola does not surround the cone, it is evident that its curve does not include a space.

Cor.

Cor. 2. An indefinite number of straight lines parallel to vB may be drawn in the plane of the parabola. For the common section of any plane passing through vB , and any point in the parabola, with the parabola (16. xi.) will be parallel to vB . BOOK I.

XVII.

If the cone $vABC$ be cut by a plane, and if the section $DKLH$, formed by the plane and the cone, surround the cone, and is not a circle, it is called an *Ellipse*. Fig. 16.

XVIII.

If the opposite cones $vABE$, $vMNE$, be cut by a plane vBE passing through the vertex v , and if they be also cut by a plane parallel to vBE , forming with the opposite cones the sections FDC , QRS ; each of the sections FDC , QRS is called an *Hyperbola*, and when mentioned together they are called *Opposite Hyperbolas*. Fig. 17.

XIX.

The plane vBE is called the *Vertical Plane* to the Hyperbola, or Opposite Hyperbolas.

Cor. 1. It is evident, as each of the opposite cones may be indefinitely extended, that an hyperbola may be indefinitely extended; and that its curve does not include a space.

Cor. 2. An indefinite number of straight lines parallel to vB or vE may be drawn in the plane of the opposite hyperbolas. For the common section of any plane passing through vB , or vE , and any point in either hyperbola, with the plane of the hyperbolas, will be parallel (16. xi.) to vB or vE .

XX.

A straight line in the plane of a conic section, which meets the curve, and which being produced both ways falls without it, is called a *Tangent*; but a straight line

BOOK I. which meets the curve of a conic section in two points, or each of the opposite hyperbolas in one, is called a *Secant*.

SCHOLIUM.

Although only the Parabola, Ellipse, and Hyperbola, are denominated Conic Sections, the attentive reader will readily perceive from the foregoing Propositions and Definitions, that five different Sections may be formed by the intersection of a cone and a plane varying its position. For if a straight line parallel to the base be within the cone and remain fixed, and a plane move about it as an axis, when the plane passes through the vertex, the intersection of the cone and plane will be a triangle, as in the first Proposition. When the plane has moved from the vertex, but still cuts both the opposite cones, the section formed in each will be an hyperbola, as in the eighteenth Definition. When the plane, proceeding in its motion round the fixed straight line, has arrived at a position parallel to that of a plane touching the cone in one of its sides, the section which it then forms with the cone is a parabola, as in the fifteenth Definition. In any other position of the moving plane, besides those already mentioned, an ellipse or circle will be formed with the cone, according to the circumstances stated in the seventeenth Definition, and in the second and fourth Propositions.

PROP. VI.

One straight line, and one only, can be drawn to touch a conic section in a given point in the curve.

Fig. 14.

Let $G D H$ be a conic section, and let D be a given point in the curve; through D one straight line, and only one, can be drawn to touch the section.

Let

Let v be the vertex of the cone $v A C B$, and through BOOK I.
 d draw $v c$ a side of the cone, meeting the base in the
 point c . Draw $c f$ (17. iii.) touching the base, and
 through $v c$, $c f$ let a plane pass, and let its line of
 common section with the plane of the section $G D H$ be
 $d e$. Then the straight line $d e$ touches the section
 $G D H$, and no other straight line can touch it in d .

For, as $c f$ meets the base in the point c only, it is
 evident from the first Definition, that every straight
 line, excepting $v c$, drawn from v to the tangent $c f$
 will fall without the superficies $v A C B$. The plane
 passing through $v c$, $c f$, therefore, can only meet the
 superficies in the straight line $v c$, and the curve of the
 section $G D H$ in the point d only. Consequently as
 $d e$ is in the plane passing through $v c$, $c f$, $d e$ touches
 the section $G D H$, according to the twentieth Defi-
 nition.

But no other straight line can touch the section
 $G D H$ in the point d . For, if it be possible, let $d i$
 touch the section, and then as $d i$ meets the curve
 $G D H$ only in the point d , it can meet the superficies
 in that point only, and it will therefore touch the su-
 perficies in d . Moreover as no straight line, except-
 ing $d e$ the line of common section, can be in the plane
 of the section $G D H$ and also in the plane $v c f$, and as
 $d i$, according to hypothesis and the twentieth Defi-
 nition, is in the plane of the section $G D H$, $d i$ is not in
 the plane $v c f$. Let a plane be passed through the
 straight lines $v d c$, $d i$, and let it cut the plane of the
 base in the straight line $k c$. Then as the straight line
 $d i$ touches the superficies, every point in it, excepting
 d , falls without the superficies, according to the tenth
 Definition. It is therefore evident, from the first De-
 finition, that every straight line drawn from v , except-
 ing $v d c$, in the plane passing through $v d c$, $d i$, will

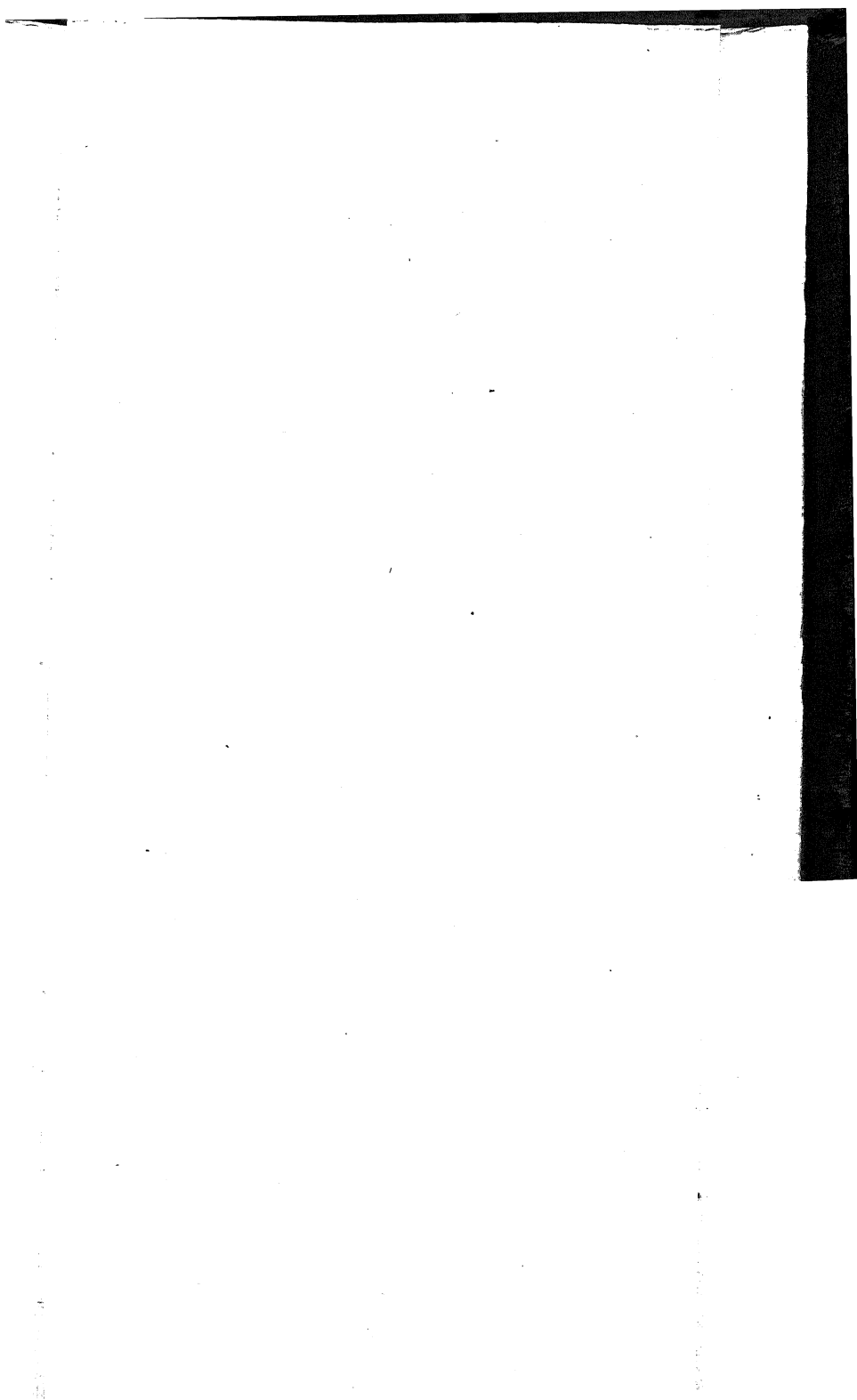
BOOK I. fall without the superficies. The plane passing through $\nu d c$, $d i$ will therefore meet the superficies in the straight line $\nu d c$ only; and consequently κc will touch the base $a c b$. Again, as the planes $\nu c \kappa$, $\nu c f$, cut one another in the straight line νc , the straight lines κc , $c f$ are not in the same straight line. The two straight lines κc , $c f$ therefore touch the circle $a c b$ in the point c , which (16. iii.) is absurd. Consequently no other straight line, besides $d e$, can be drawn to touch the section $g d h$ in the point d .

Cor. 1. If a straight line as $c f$ touch the base of the cone in the point c , and from ν , the vertex, the side νc be drawn; a plane passing through $c f$, $c \nu$ will touch the conical superficies in the side νc ; and it is evident from the first Definition (and 1. xi.) that this plane produced on the other side of ν will touch the opposite superficies in $c \nu$ produced.

Cor. 2. If a straight line as $d e$ touch the conical superficies, or a conic section $g d h$, and a side $\nu d c$ of the cone be drawn through d the point of contact, a plane passing through this side of the cone and the tangent $d e$ will touch the superficies of the cone; and being produced beyond ν , it will touch the opposite superficies in νc produced. This is evident from the demonstration of the Proposition, and the preceding Cor.

Cor. 3. If the section $g d h$ be an hyperbola, the tangent $d e$ cannot meet the opposite hyperbola. For $d e$ is the common intersection of the plane $\nu c f$ and the plane of the section $g d h$, and, by the first Cor. the plane $\nu c f$ touches the opposite superficies in $c \nu$ produced. It is therefore evident from the eighteenth Definition that the tangent $d e$ cannot meet the opposite hyperbola.

PROP.



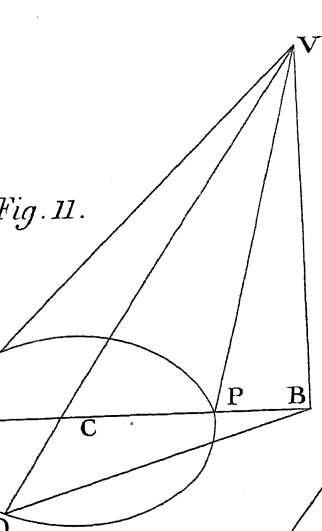


Fig. 11.

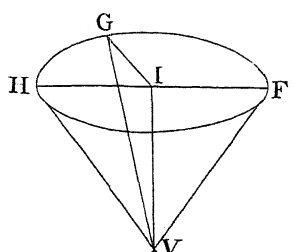


Fig. 10.

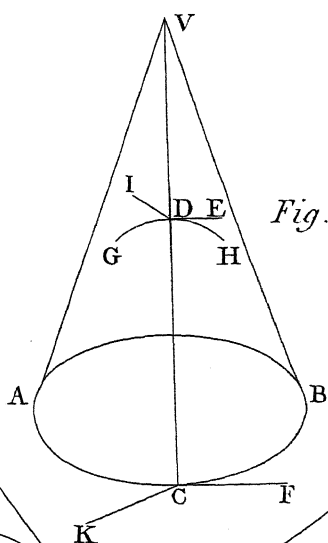


Fig. 14.

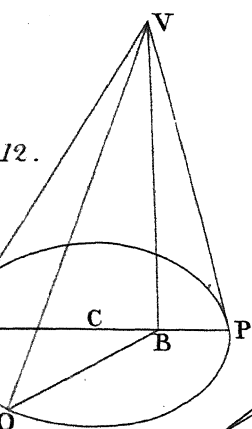


Fig. 12.

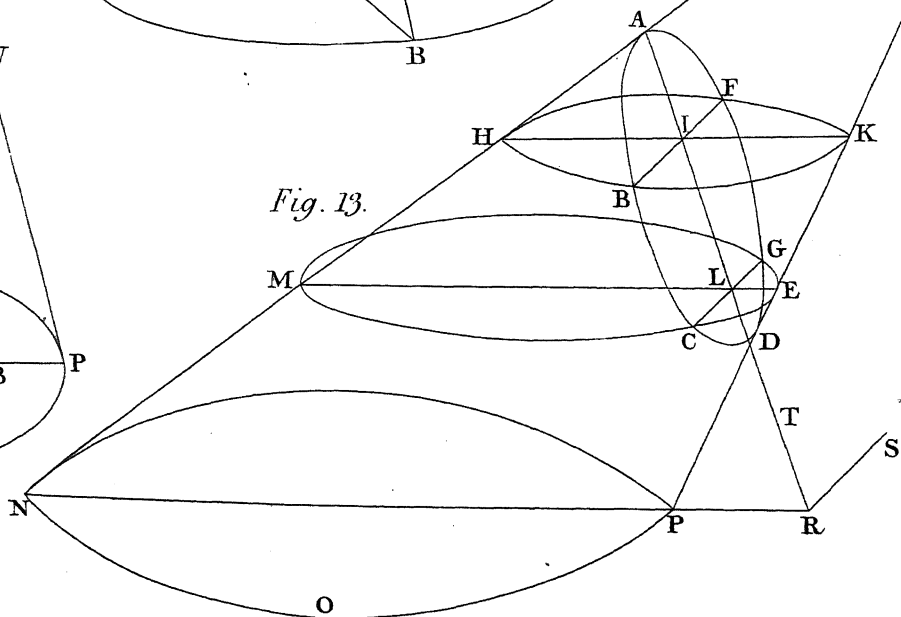
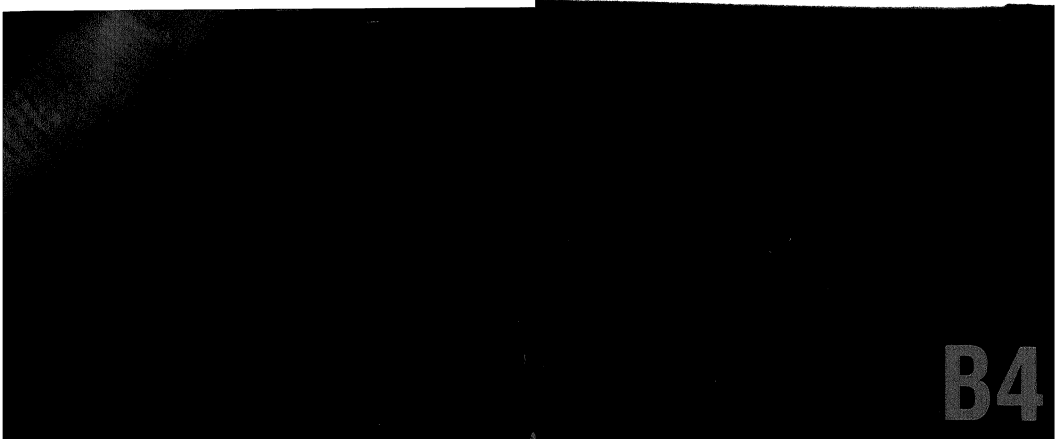
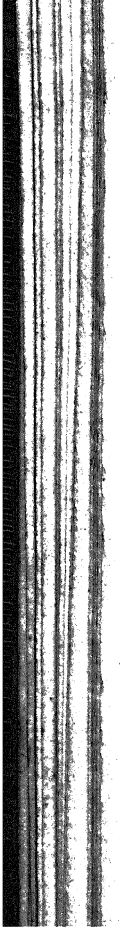


Fig. 13.



B4

PROP. VII.

BOOK
I.

If a straight line pass through the vertex and fall without the opposite cones, two planes, and only two, can be drawn through it to touch the conical superficies; and these planes will be on the opposite sides of a plane passing through the straight line, and cutting the base.

Let the straight line VG pass through V the vertex, and fall without the opposite cones $VAMB$, DVE ; two planes, and only two, can be drawn through it to touch the superficies; and if a plane pass through VG , and cut the base in the straight line CF , the tangent planes will be on the opposite sides of the plane passing through VG , CF . Fig. 18.
and
19.

First let the straight lines VG , CF be parallel. Let CF be bisected in I , and draw ILM at right angles to CF , and let it meet the circumference of the circle in the points M , L . Let a plane pass through the straight line VG and the point I , and this plane will touch the superficies. For let it cut the plane of the base in the straight line IK . Then as CF , VG are parallel, and as the plane of the base passes through CF , and the plane VIK passes through VG , the intersection IK of these planes will be parallel to CF , by the second Lemma. But as IM bisects CF at right angles, it passes through the center of the circle (Cor. 1. iii.), and as IK , CF are parallel, and as the angle ILF is a right one, the angle KIL is also a right one (29. i.), and therefore IK (16. iii.) touches the circle in I . Consequently, by Cor. 1. Prop. VI. the plane passing through VI , IK , or, as above, through VG , VI , touches the superficies; and in the same manner it may be proved that the plane passing through VG and the point M touches the superficies. Fig. 18.

Secondly, let the straight line VG be not parallel to Fig. 19.

 CF CF

BOOK CF, but let it meet it in K. Draw KI, KM (17. iii.)
 I. touching the base in I, M; and then the plane passing through VG, IK, and also that passing through VG and MK, will touch the superficies, by Cor. 1. Prop. VI.

It is evident, in either case, that no other plane, besides the above-mentioned two, can pass through VG, and touch the superficies; and that one of these tangent planes is on the one side, and the other on the opposite side of the plane passing through VG, CR.

PROP. VIII.

If a conic section surround the cone, two straight lines, and only two, parallel to one another, can be drawn to touch the section; if the section does not surround the cone, no straight line, parallel to a tangent, can be drawn to touch the section; but if the section be an hyperbola, one straight line, and one only, parallel to a tangent, can be drawn to touch the opposite hyperbola.

Fig. 16.

Part I. Let the section D H L K surround the cone V A F B, and let the straight line GD touch the section in the point D: another straight line, and only one, parallel to GD, can be drawn to touch the section.

For let V D A be the side of the cone passing through D, the point of contact, and let a plane pass through V A, DG, and this plane will touch the conical superficies in the side V A, by Cor. 2. Prop. vi. In this plane, and through V, the vertex of the cone, draw VT parallel to DG. Then VT will fall without the opposite cones; and by Prop. VII. another plane can be passed through VT touching the conical superficies. Let this plane touch the superficies in the side V L B, and let its intersection with the plane of the section D H L K be LI. Then as LI is in the plane touching the

BOOK
I.

the cone in the side VLB , it meets the conical superficies in the point L only. It will therefore meet the curve of the section $DHLK$ in the point L only, and consequently it will touch the section: and as the plane $TVLI$ passes through VT , and the plane of the section passes through DG parallel to VT , by the second Lemma LI is parallel to DG . And as no other plane passing through VT can touch the conical superficies, besides the two $TV A$, $TV B$, it is evident, from the second Lemma, that no other straight line besides LI , parallel to DG , can be drawn to touch the section $DHLK$.

Part II. Let FDC be a section which does not surround the cone, and let DG touch the section in the point D . No other straight line, parallel to DG , can be drawn to touch the section.

Fig. 15.
and
17.

For let VBE be the vertical plane to the parabola, or hyperbola, as in the fifteenth, sixteenth, eighteenth, and nineteenth Definitions. Let VDA be the side of the cone passing through D , the point of contact; and through VDA , DG let a plane pass, and let it cut the vertical plane in the straight line VT . Then, by Cor. 2. Prop. VI. the plane $V DG$ will touch the conical superficies, and (16. xi.) DG , VT will be parallel; and as VT is in the plane, touching the conical superficies in the side VDA , it will fall without the opposite cones. Another plane, therefore, and only one, can be passed through VT to touch the conical superficies, by Prop. VII. But when the section is a parabola, the other plane passing through VT and touching the superficies is the vertical plane VBE , which is parallel to the parabola. When the section is an hyperbola, then the vertical plane VBE passes through VT , and cuts the base of the cone in the straight line BE ; and supposing TVL to be the other plane passing through VT ,
and

BOOK and touching the conical superficies, the planes $T V L$,
I. $V D G$ are on opposite sides of $V B E$, by the seventh Proposition. Consequently the plane $T V L$ cannot meet the hyperbola $F D C$. It therefore follows from the above, and the second Lemma, that if the section does not surround the cone, no straight line, parallel to a tangent, can be drawn to touch the section.

Fig. 17.

Part III. Let $F D C$, $a r s$ be opposite hyperbolas, and let $D G$ touch the hyperbola $F D C$ in the point D . Then one straight line, and only one, parallel to $D G$ can be drawn to touch the opposite hyperbola $a r s$.

For, every thing remaining as in the preceding part, through the sides $V A$, $V L$, in which the planes $T V A$, $T V L$, passing through $T V$ parallel to $D G$, touch the superficies, let a plane be passed, cutting the vertical plane in the straight line $V W$ and the plane of the hyperbolas in the straight line $D R$. Then as $V W$, $R D$, $L V$ are in the same plane, and as (16. XI.) $V W$, $R D$ are parallel, and $L V$ meets $V W$, it will also meet $R D$, by the third Lemma. Let them meet in the point R . Then as the plane $T V L$ touches the opposite cone $M V N$ in $L V$ produced, it will meet the plane of the hyperbolas in the point R . Let the intersection of these two planes therefore be $R X$; and as the plane of the hyperbolas passes through $D G$, and the plane $T V L$ passes through $T V$ parallel to $D G$, by the second Lemma $R X$ is parallel to $D G$; and being in the plane touching the conical superficies, it will touch the hyperbola $a r s$ in the point R . It is also evident, for the same reasons as are mentioned above, that no other straight line parallel to $R X$, or $D G$, can be drawn to touch the hyperbola $a r s$.

Cor. I. If a straight line touch a conic section, a straight line drawn through any point within the section, and parallel to it, will meet the curve in two points.

points. For let every thing remain as in the demonstration of the Proposition, and let p be any point within the section. Through v t and the point p let a plane be passed, and let this plane cut the plane of the section in the straight line h k ; and, by the second Lemma, h k will be parallel to g d the tangent, and also to v t . Now as the point p is within the section it is also within the cone, and therefore, by the first Proposition, the plane passing through v t and the point p will cut the cone in two sides; and as these two sides and v t , h k are in the same plane, and v t , h k are parallel, h k will meet each of these two sides, by the third Lemma; and as h k is in the plane of the section, it must meet the curve of the section in the same points in which it meets these two sides of the cone. It is also evident from the Proposition, and the preceding part of this Corollary, that a straight line drawn through any point within the opposite hyperbola a r s , and parallel to g d , will meet the curve a r s in two points.

Cor. 2. If a straight line meet the curve of a conic section in two points, two straight lines may be drawn parallel to it to touch the section, if it surround the cone; but if the section does not surround the cone, only one straight line parallel to a secant can be drawn to touch the section; and, if the section be an hyperbola, only one straight line parallel to a secant can be drawn to touch the opposite hyperbola. For, let h , k be the points in which the secant h k meets the curve of the section; and through h k and v , the vertex of the cone, let a plane be passed, and, if the section surround the cone, draw v t in this plane parallel to h k . Then as this plane, by the Cor. to Prop. I. can only cut the opposite superficies in straight lines drawn through v the vertex and the points h , k , it is evident,

BOOK
I.Fig. 15, 16,
and
17.

Fig. 16.

BOOK
I.

Fig. 15.
and
17.

dent that VT must fall without the opposite cones. Consequently, by Prop. VII. two planes can be passed through VT to touch the conical superficies, one on each side of HK ; and the intersections of these planes with the plane of the section will touch the section, and, by the second Lemma, these tangents will be parallel to HK . If the section does not surround the cone, let the plane passing through the secant HK and V cut the vertical plane VBE in the straight line VT , and, for the same reasons as are mentioned above, VT will fall without the opposite cones. Then through VT two planes may be passed, by Prop. VII. touching the conical superficies. But, according to the demonstration of the Proposition, only one of these planes can meet the parabola, and one of them can meet the hyperbola, and the other the opposite hyperbola; and, by the second Lemma, GD , the intersection of the plane $TV A$ with the plane of the section FDC , will be parallel to HK , and RX the intersection of the plane $TV L$ with the plane of the opposite hyperbolas, will also be parallel to HK , and each of the straight lines GD , RX must touch the section which it meets.

PROP. IX.

If a straight line meet the curve of a conic section in two points, any straight line parallel to it, drawn through a point within the same section, or, if the section be an hyperbola, within the opposite hyperbola, will also meet the curve of the section, in which it is drawn, in two points. And if a straight line meet each of the curves of two opposite hyperbolas in one point, a straight line parallel to it, drawn through any point in the plane of these sections, will also meet each of the curves of these opposite hyperbolas in one point.

Fig. 15.

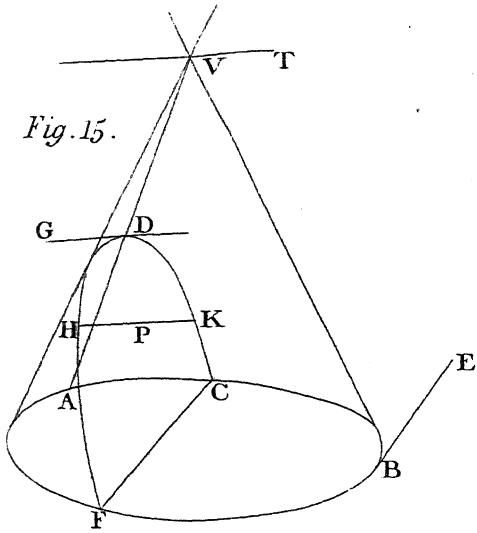


Fig. 17.

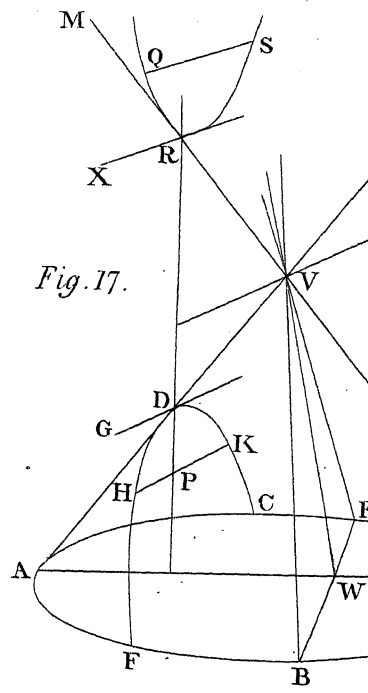


Fig. 16.

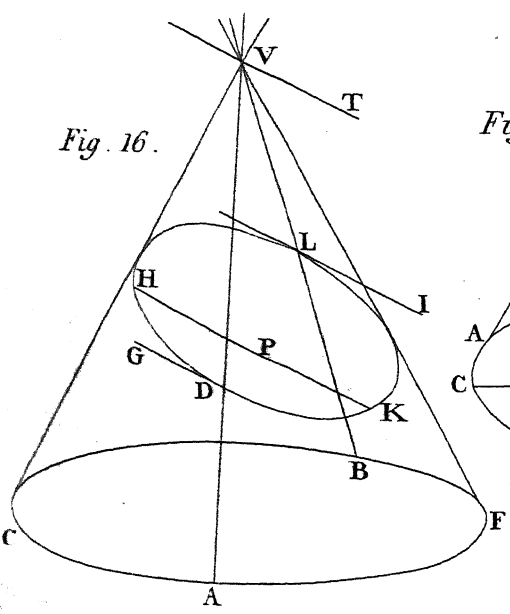
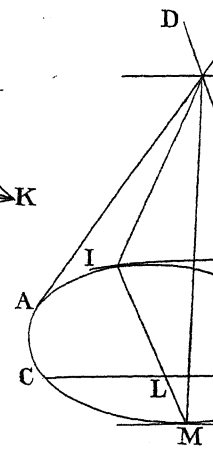
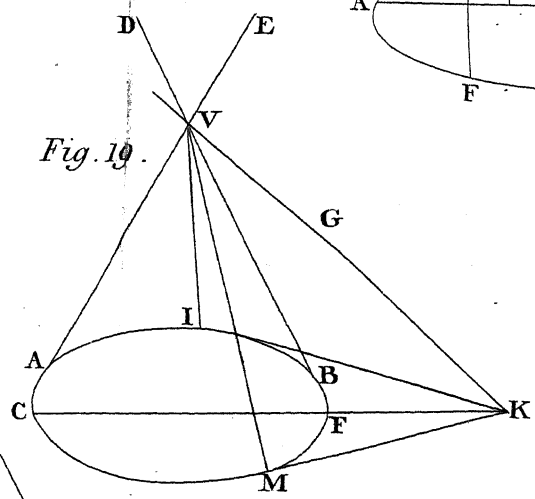


Fig. 19.



28

E

—

Part I. Let $A C D B$ be a conic section, and let the straight line $c d$ meet the curve in the points c, d ; a straight line, as $A B$, drawn parallel to $c d$ through any point P within the section will meet the curve in two points. BOOK
I.
Fig. 20.

For let v be the vertex of the cone $v E G H F$, in which the section is formed, and through $c d$ and v let a plane be passed, and let it cut a plane passing through $A B$ and v in the straight line $v T$. Then the plane passing through $c d$ and v must cut the cone in the sides $v c, v d$, and, by the second Lemma, $v T$ is parallel to $c d$ and also to $A B$. Again, as $v T$ is in the plane $v c d$, and as this plane, by Cor. Prop. I. cuts the opposite superficies only in $v c, v d$, or in these lines produced, it follows that $v T$ falls without the opposite cones. The plane passing through $v T, A B$ will therefore cut the cone in two sides, and, by the third Lemma, $A B$ will meet each of these two sides in the superficies of the cone, and being in the plane of the section, it must meet the curve in the same points. If the section be an hyperbola, it is evident, for the same reasons, that a straight line drawn parallel to $c d$, through any point within the opposite hyperbola will meet the curve in two points.

Part II. Let $A c, B d$ be two opposite hyperbolas, and let the straight line $c d$ meet the curve of one of them in the point c , and the curve of the other in the point d . Through the point P in the plane of these sections let the straight line $A B$ be drawn parallel to $c d$; $A B$ will meet each of the opposite hyperbolas $A c, B d$ in one point. Fig. 21.

For let v be the vertex of the opposite cones, in which the hyperbolas are formed, and through $c d$ and v let a plane be passed, and let it cut a plane passing through $A B$ and v in the straight line $v T$. Then
the

BOOK the plane passing through $c d$ and v must cut the opposite cones in the sides $v c$, $v d$; and, by the second
I. Lemma, $v t$ is parallel to $c d$ and also to $a b$. Again, as $v t$ is in the plane $v c d$, and as this plane, by Cor. Prop. I. cuts the opposite superficies only in $v c$, $v d$, or in these lines produced, it follows that $v t$ falls within the opposite cones. The plane passing through $v t$, $a b$ will therefore cut the opposite cones in two sides; and from the above, and the third Lemma, $a b$ will meet one of these two sides in the one superficies, and the other in the opposite superficies. But as $a b$ is in the plane of the opposite hyperbolas $a c$, $b d$, it must meet the curve of one of the hyperbolas and the superficies, in which this hyperbola is, in the same point. The straight line $a b$ will therefore meet the curve of each of the opposite hyperbolas in one point.

Cor. If a straight line as $c d$ meet the curve of a conic section in two points, it will fall wholly within the section, but being produced it will fall without the section. If a straight line meet each of the curves of opposite hyperbolas in one point, it will fall wholly without the hyperbolas, but being produced it will fall on the one side within one hyperbola, and on the other within the opposite hyperbola. For it is evident from the demonstration of the Proposition that $c d$ in Fig. 20. is within the cone, and that being produced it must fall without it; and in Fig. 21. it is evident that $c d$ is without the opposite cones, and that being produced it falls on the one side within one cone, and on the other side within the opposite cone.

PROP. X.

If a straight line cut either or both of the opposite conical superficies, and meet a straight line which is parallel to the base of the cone, and which cuts either of the opposite super-

superficies, the rectangle under the segments of the first mentioned line will be to the rectangle under the segments of the other in the same ratio, wherever the point of concurrence may be in the first mentioned line.*

BOOK
I.

Let the straight line FGH cut either or both of the opposite superficies in the points G, H , and meet, in the point F , the straight line FDE parallel to the base of the cone, and cutting either of the opposite superficies in the points E, D ; the rectangle under GF, FH will be to the rectangle under DF, FE in the same ratio, wherever the point of concurrence F may be in the straight line FGH .

Fig. 26,
27, 28.

Case 1. If the straight line FGH be also parallel to the plane of the base, then the section $G D H E$, formed by the cone and the plane passing through FGH, FED will be a circle, by the fourth Lemma, and Prop. II. and therefore (35 or Cor. 36. iii.) the rectangle under GF, FH will be to the rectangle under DF, FE in the ratio of equality.

Fig. 26.

Case 2. Let FGH be not parallel to the base of the cone. Through FED let a plane be passed parallel to the base, and let the section formed by it with the cone be the circle $DEIK$, as in the second Proposition. Through the points G, H , and V , the vertex of the cone, let a plane be passed, and let it cut the superficies in the sides AVI, BVK , the plane of the base in the straight line ABL , and the plane of the circle $DEIK$ in the straight line IFK ; and in the plane VGH draw VL parallel to GH , and let it meet ABL in the point L .

Fig. 27,
28.

* When two secants meet one another, the segments of either of the two are its parts between the point of concurrence and the points in which it meets the superficies; and if a tangent meet a secant, or another tangent, its magnitude is limited by the point of concurrence, and its point of contact.

Then

BOOK I. Then as the straight lines ABL , IFK (16. xi.) are parallel to one another, and the straight lines VL , FGH also parallel to one another, in the triangles VLB , HFK , the angles (29. i.) BVL , VLB in the one are equal to the angles KHF , HKF in the other, each to each; and in the triangles VLA , GFI , the angles VAL , AVL in the one are equal to the angles GIF , IGF in the other, each to each. The triangles VLB , HFK are therefore equiangular to one another, as are also the triangles VLA , GFI to one another. Hence (4. vi.)

$$VL : LA :: GF : FI, \text{ and}$$

$VL : LB :: HF : FK$, and therefore, by the fifth Lemma, $VL^2 : AL \times LB :: GF \times FH : IF \times FK$. But (35 or Cor. 36. iii.) $IF \times FK = DF \times FE$, and consequently,

$$VL^2 : AL \times LB :: GF \times FH : DF \times FE.$$

Cor. 1. If the straight line FGH meet the straight line FP parallel to the base, and touching either superficies in the point P , the rectangle under GF , FH will be to the square of FP in the same ratio, wherever the point of concurrence may be in the line FGH , as is evident from the above (and 36. iii.) And if the straight line MV , passing through V the vertex, be parallel to the base, and meet GH in M , MV is to be considered as a tangent; for as above, by similar triangles,

$$VL : LA :: GM : MV, \text{ and}$$

$VL : LB :: HM : MV$, and therefore, by the fifth Lemma, $VL^2 : AL \times LB :: GM \times MH : MV^2$.

Cor. 2. If a straight line cut either or both the opposite superficies, and meet a straight line parallel to the base, and touching or cutting either superficies; the rectangle under the segments of the first mentioned line will be to the square of the tangent, or the rectangle under the segments of the secant which it meets
in

in a constant ratio, wherever the point of concurrence may be in the first mentioned line. For this ratio will be either that of equality, as in the first case of the demonstration, or it will be that of VL^2 to $AL \times LB$, as in the second case, and in the preceding Cor.

PROP. XI.

If a straight line touch either of the opposite superficies, and meet a straight line parallel to the base of the cone, and which cuts either of the opposite superficies, the square of the tangent will be to the rectangle under the segments of the secant in the same ratio, wherever the point of concurrence may be in the tangent.

Let the straight line TF touch either of the opposite superficies in the point T , and meet in the point F the straight line FE , parallel to the base of the cone, and cutting either of the opposite superficies in the points G, E ; the square of the tangent TF will be to the rectangle under GF, FE , in the same ratio, wherever the point F may be in the tangent TF . Fig. 22.
23.

Case 1. If the tangent TF be also parallel to the plane of the base, then the section TGE , formed by the cone and the plane passing through TF, FE will be a circle, by the fourth Lemma, and Prop. II, and the square of TF (36. iii.) will be to the rectangle GF, FE in the ratio of equality. Fig. 22-

Case 2. Let TF be not parallel to the base of the cone. Through FE let a plane be passed parallel to the base, and let the section formed by it with the cone be the circle DGE , as in the second Proposition. Through the tangent TF and V , the vertex of the cone, let a plane be passed, and let it touch the superficies in the side VRT , according to Cor. 2. Prop. VI. and let it cut the plane of the base in the straight line CI , and Fig. 23-

D the

BOOK the plane of the circle DGB in the straight line DF .
I. In the plane VCL draw VL parallel to TF , and let it meet the base in the point L . Then as the straight lines CL, DF (16. xi.) are parallel to one another, and the straight lines VL, TF also parallel to one another, in the triangles VCL, TDF , the angles (29. i.) CVL, VCL in the one are equal to the angles DTF, TDF in the other, each to each.

The triangles VCL, TDF are therefore equiangular, and consequently (4. vi.)

$$VL : LC :: TF : FD, \text{ and by the fifth Lemma, } VL^2 : LC^2 :: TF^2 : FD^2.$$

But (36. iii.) FD^2 is equal to the rectangle, under GF, FE , and consequently $VL^2 : LC^2 :: TF^2 : GF \times FE$.

Cor. 1. If a straight line, as TF , touch either of the opposite superficies in T , and meet a straight line, as FD , parallel to the base of the cone, and which touches either superficies as in D ; the square of TF will be to the square of FD in the same ratio, wherever the point of concurrence F may be in TF . If MV , passing through V the vertex, be parallel to the base and meet TF in M , then MV is to be considered as a tangent; and as above $VL^2 : LC^2 :: TM^2 : MV^2$.

Cor. 2. If a straight line touch either of the opposite superficies, and meet a straight line parallel to the base, and touching or cutting either superficies; the square of the first mentioned tangent will be to the square of the tangent, or the rectangle under the segments of the secant which it meets in a constant ratio, wherever the point of concurrence may be in the first mentioned tangent. For this ratio will be that of equality, as in the first case of the demonstration, or it will be that of VL^2 to LC^2 , as in the second case.

PROP.

PROP. XII.

BOOK
I.

If the first of two straight lines be parallel to the second and touch or cut either or cut both of the opposite superficies, and if the second also touch or cut either, or cut both of the opposite superficies, and if each of the two meet a straight line parallel to the base of the cone and touching or cutting either superficies; then the square of the first, if a tangent, or the rectangle under its segments, if a secant, will be to the square of the tangent or the rectangle under the segments of the secant which it meets, as the square of the second, if a tangent, or the rectangle under its segments, if a secant, to the square of the tangent, or the rectangle under the segments of the secant which it meets.

Case 1. If the first and second straight lines, parallel to one another, be also parallel to the plane of the base, then the square of the first, if a tangent, or the rectangle under its segments, if a secant, will be to the square of the tangent, or the rectangle under the segments of the secant which it meets in a ratio of equality, as in the first case of the demonstration of Prop. X. and XI. And, for the same reasons, the square of the second, if a tangent, or the rectangle under its segments, if a secant, will be to the square of the tangent, or the rectangle under the segments of the secant which it meets in a ratio of equality. In this case therefore the Proposition is evident.

Case 2. Let the first and second straight lines, parallel to one another, be not parallel to the plane of the base. Suppose a plane to pass through each of the two parallel lines, and v the vertex, and then these planes will cut one another, and their line of common section will pass through v , and, by the second Lemma, it will be parallel to each of the two straight lines parallel to

BOOK one another. Let VL be their line of common section,
 I. as in Fig. 27. 28. and 23. and let it meet the base in the point L . Then the planes passing through the two parallel lines and VL , will cut the plane of the base in straight lines passing through L ; and each of these lines of common section with the plane of the base will cut or touch the base, as in the second case of the demonstration of the tenth and eleventh Propositions; and the rectangle under the segments of either, if a secant, or its square, if a tangent, will be (35 and 36. iii.) equal to the rectangle under AL , LB , LA being a straight line cutting the base in the points B , A . Consequently by Cor. 2. Prop. X. and Cor. 2. Prop. XI. the square of the first straight line, if a tangent, or the rectangle under its segments, if a secant, will be to the square of the tangent, or the rectangle under the segments of the secant, which it meets as the square of VL to the rectangle under AL , LB . For the same reason, the square of the second, if a tangent, or the rectangle under its segments, if a secant, will be to the square of the tangent, or the rectangle under the segments of the secant, which it meets as the square of VL to the rectangle under AL , LB . Hence (11. v.) if the first of two straight lines be parallel, &c.

SCHOLIUM.

As every point in the curve of a Conic Section is also in the conical superficies, it is evident that all the Propositions demonstrated concerning straight lines touching or cutting the conical superficies, or opposite superficies, may be transferred to straight lines, which in the same manner touch or cut a conic section, or opposite hyperbolas.

PROP.

PROP. XIII.

BOOK
I.

If there be four straight lines in the plane of a conic section, and if AB the first meet CB the second, and DE the third meet FE the fourth, and if the first be parallel to the third, and the second to the fourth, and if each of them either touch or cut a conic section, or cut opposite hyperbolas; then the square of AB, if a tangent, or the rectangle under its segments, if a secant, will be to the square of CB, if a tangent, or the rectangle under its segments, if a secant, as the square of DE, if a tangent, or the rectangle under its segments, if a secant, to the square of FE, if a tangent, or the rectangle under its segments, if a secant.

Fig. 24.

Case 1. If the straight lines AB, CB, and consequently DE, FE, be each parallel to the base of the cone in which the section was formed, or the base of the opposite cone, the section must be a circle, as in the first case of Prop. X. or Prop. XI, and the ratio above stated will be that of equality.

Case 2. Let AB, DE be not parallel to the base of the cone, in which, or in which and its opposite, the section or opposite hyperbolas were formed; but let CB, FE be parallel to the base, and then the Proposition is evident from the twelfth Proposition.

Case 3. Let neither AB nor CB, and consequently neither DE nor FE, be parallel to the base of the cone; but suppose BG, EH to be straight lines parallel to the base of the cone in which the section, or in which and the opposite cone the opposite hyperbolas were formed; and let BG, EH touch or cut either of the opposite conical superficies. Then by the twelfth Proposition, the square of AB, if a tangent, or the rectangle under its segments, if a secant, will be to the square of BG, if a tangent, or the rectangle under its segments, if a secant,

BOOK cant, as the square of DE , if a tangent, or the rectangle under its segments, if a secant, to the square of EH , if a tangent, or the rectangle under its segments, if a secant. Again, by the twelfth Proposition and inversion, the square of BG , if a tangent, or the rectangle under its segments, if a secant, is to the square of CB , if a tangent, or the rectangle under its segments, if a secant, as the square of EH , if a tangent, or the rectangle under its segments, if a secant, to the square of FE , if a tangent, or the rectangle under its segments, if a secant. Consequently,

$$\left. \begin{array}{c} t. AB^2 \\ \text{OR} \\ f. AB^2 \end{array} \right\} : \left\{ \begin{array}{c} t. BG^2 \\ \text{OR} \\ f. BG^2 \end{array} \right\} : \left\{ \begin{array}{c} t. CB^2 \\ \text{OR} \\ f. CB^2 \end{array} \right\}$$

$$\left. \begin{array}{c} t. DE^2 \\ \text{OR} \\ f. DE^2 \end{array} \right\} : \left\{ \begin{array}{c} t. EH^2 \\ \text{OR} \\ f. EH^2 \end{array} \right\} : \left\{ \begin{array}{c} t. FE^2 \\ \text{OR} \\ f. FE^2 \end{array} \right\}$$

The square of AB therefore (22. v.) if a tangent, or the rectangle under its segments, if a secant, is to the square of CB , if a tangent, or the rectangle under its segments, if a secant, as the square of DE , if a tangent, or the rectangle under its segments, if a secant, to the square of FE , if a tangent, or the rectangle under its segments, if a secant.

Cor. If AB, CB, DE, FE be tangents, then it is evident (22. vi.) that $AB : CB :: DE : FE$.

PROP. XIV.

Any straight line parallel to the side of a cone, provided it be not in the plane touching the cone in that side, will meet one of the opposite superficies in one point, and in one point only.

Fig. 29. Let the straight line DC be parallel to VB a side of
 30. the cone $VAMB$, but not situated in the plane touching
 ing

ing the cone in the side vB ; the straight line DC will meet one of the opposite superficies in one point, and in one point only. BOOK
I.

Let a plane pass through the parallels DC , vB , and as by hypothesis DC is not in the plane touching the cone in the side vB , the plane passing through DC , vB must cut the cone. Let it cut the opposite superficies therefore in the straight lines AV , BV . Then as AV meets BV in v the vertex, and as it is in the same plane with the parallels vB , DC , by the third Lemma AV , or AV produced, must also meet DC . Let them meet in D . Then DC must meet one of the superficies in D , and as it is parallel to vB , it is evident it cannot meet the other superficies. It is also evident, that it can meet one of the opposite superficies in one point only; for on one side of D it is entirely within the superficies, and on the other entirely without it.

PROP. XV.

If a straight line parallel to a side of the cone cut either of the opposite conical superficies, and meet two straight lines parallel to the base of the cone, and which cut either superficies; the segments of the first mentioned line, between the superficies and the points of concurrence, will be to one another as the rectangles under the segments of the secants which it meets.

Let the straight line DC , parallel to vB a side of the cone $vAMB$, cut either of the opposite superficies in the point D , and meet in the points E , R the straight lines EIF , LRT , which are parallel to the base of the cone, and cut either superficies in the points I , F , and L , T ; then the segment DE is to the segment DR as the rectangle under EI , EF to the rectangle under LR , RT .

Fig. 29.
30.

BOOK For through the parallels DC, VB let a plane pass,
I. and let it cut the plane of the base in the straight line
 ACB , and the superficies in AV, BV . Through each
of the straight lines LRT, EIF let a plane pass pa-
rallel to the base AMB , and let $OLNT, IFHG$ be the
circles formed, as in the second Proposition. Let $EGH,$
 ORN be the intersections of these circles and the plane
passing through AV, BV . Let EGH meet AV in G
and BV in H ; and let ORN meet AV in O and BV in
 N . Then (16. xi.) the straight lines $EGH, ORN,$
 ACB are parallel, and therefore (34. i.) EH, CB, RN
are equal to one another; the angle DGE (29. i.) is
equal to the angle DAC , and the angle DEG to the
angle DCA . Hence (4. vi.) $DE:EG::DC:AC,$
and $DR:OR::DC:AC$; and therefore (11. v.)
 $DE:EG::DR:OR$, and (16. v.) $DE:DR::EG:OR$.
Consequently (1. vi.) $DE:DR::EG \times EH:$
 $OR \times RN$.

But (35. and 36. iii.) $EG \times EH = EI \times EF$, and
 $OR \times RN = LR \times RT$; and therefore $DE:DR::$
 $EI \times EF:LR \times RT$.

Cor. If a straight line parallel to a side of the cone
cut either of the opposite conical superficies, and meet
two straight lines parallel to the base, and which meet
either superficies; its segment between the superficies,
and the first of the two parallel to the base, will be to
its segment between the superficies and the second, as
the square of the first, if a tangent, or the rectangle
under its segments, if a secant, to the square of the se-
cond, if a tangent, or the rectangle under its segments,
if a secant. For every thing remaining as above, if
 EF be parallel to the base, and touch either superficies
in F , EF will be in the plane of the circle $IFHG$ and
(36. iii.) the square of EF will be equal to the rectan-
gle under EI, EF . And, for the same reasons, if the
point

point R were without the circle, the square of a straight line parallel to the base, drawn from R and touching either superficies would be equal to the rectangle under LR, RT . If a straight line VK parallel to the base pass through V, the vertex, and meet CD in K, VK is to be considered as a tangent. For $DK : KV :: DC : CA$, and $DR : OR :: DC : CA$. Hence (II. V.) $DK : KV :: DR : OR$, and (16. V.) $DK : DR :: KV : OR$. But KV, RN being parallel to the base are parallel to one another, and therefore (34. I.) KV, RN are equal, and consequently (I. vi.) $DK : DR :: KV^2 : OR \times RN$.

BOOK
I.

PROP. XVI.

If a straight line cutting a parabola or hyperbola be parallel to a side of the cone in which the section is formed, and if it meet two straight lines which are parallel to one another, and meet the same section, or the opposite hyperbolas; its segment between the curve and the first of the two parallels will be to its segment between the curve and the second, as the square of the first, if a tangent, or the rectangle under its segments, if a secant, to the square of the second, if a tangent, or the rectangle under its segments, if a secant.

Suppose the straight line BC to cut the curve of a parabola or hyperbola in the point A, and to be parallel to a side of the cone in which the section is formed, and let it meet the straight lines BD, CE which are parallel to one another, and meet the same section, or the opposite hyperbolas; then AB is to AC as the square of BD , if a tangent, or the rectangle under its segments, if a secant, to the square of CE , if a tangent, or the rectangle under its segments, if a secant.

Fig. 25.

For, as BC cuts the curve of the parabola or hyperbola

BOOK bola in the point A, it will cut the conical superficies
I. in which the curve is formed in the same point; and
 the straight lines B D, C E will meet the conical superficies in the same points in which they meet the curve of the section, or the curves of the opposite hyperbolas. If therefore B D, C E be parallel to the base of the cone, the Proposition is evident from the Cor. to Prop. XV. but if they are not parallel to the base of the cone, let B F, C G be parallel to the base of the cone, and let them meet the same or the opposite conical superficies. Then by the Cor. to Prop. XV. A B is to A C as the square of B F, if a tangent, or the rectangle under its segments, if a secant, to the square of C G, if a tangent, or the rectangle under its segments, if a secant. Again, by Prop. XII. (and 16. v.) the square of B F, if a tangent, or the rectangle under its segments, if a secant, is to the square of C G, if a tangent, or the rectangle under its segments, if a secant, as the square of B D, if a tangent, or the rectangle under its segments, if a secant, to the square of C E, if a tangent, or the rectangle under its segments, if a secant. Consequently (II. v.) A B is to A C as the square of B D, if a tangent, or the rectangle under its segments, if a secant, to the square of C E, if a tangent, or the rectangle under its segments, if a secant.

PROP. XVII.

If two straight lines meeting one another touch a conic section, or opposite hyperbolas, and if a secant parallel to one of them meet the other and the straight line joining the points of contact, the rectangle under the segments of the secant between the curves and the tangent will be equal to the square of its segment between the tangent and the line joining the points of contact.

Let

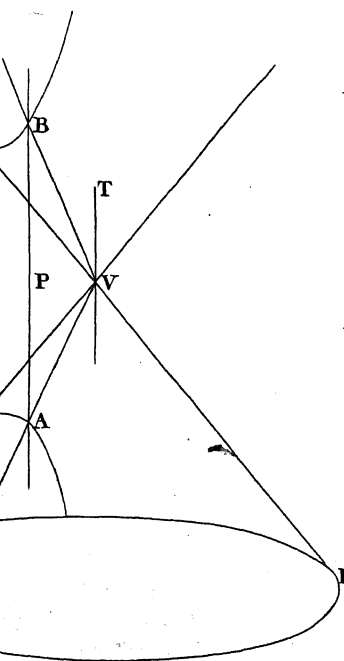
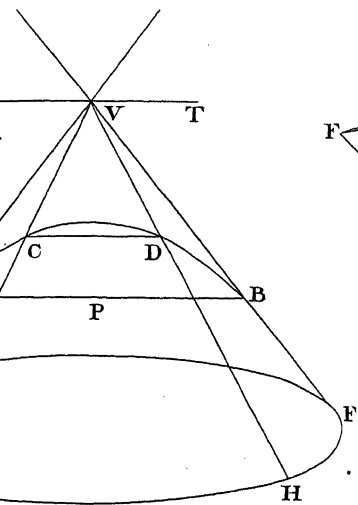


Fig. 23.

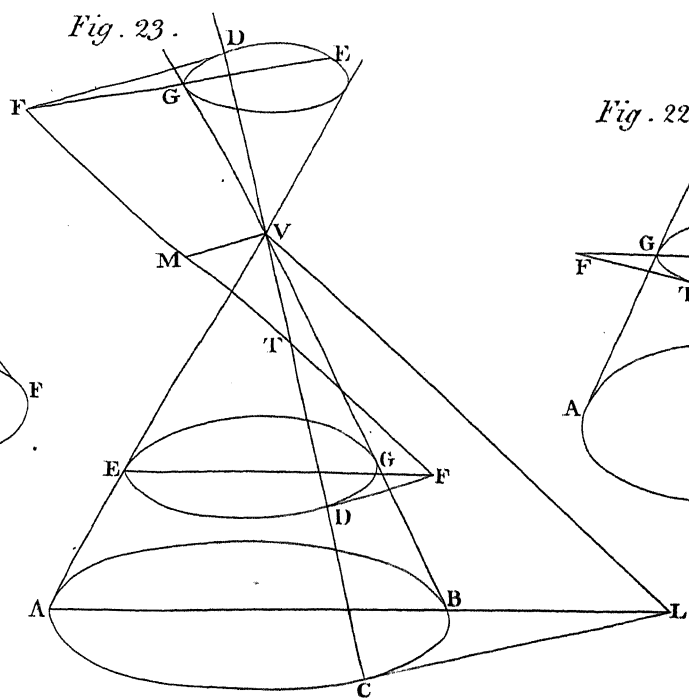


Fig. 22.

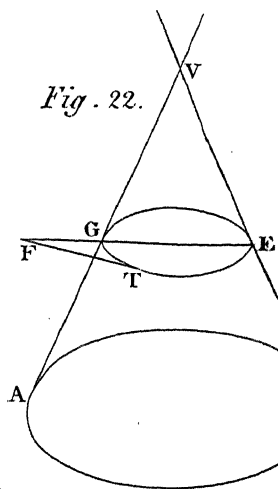


Fig. 24.

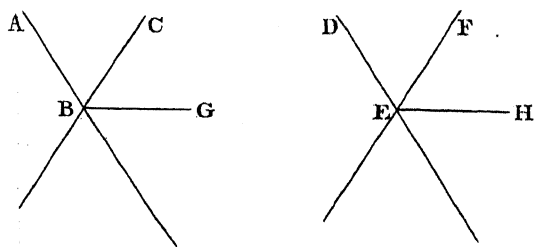
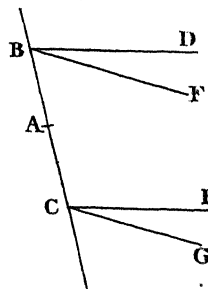
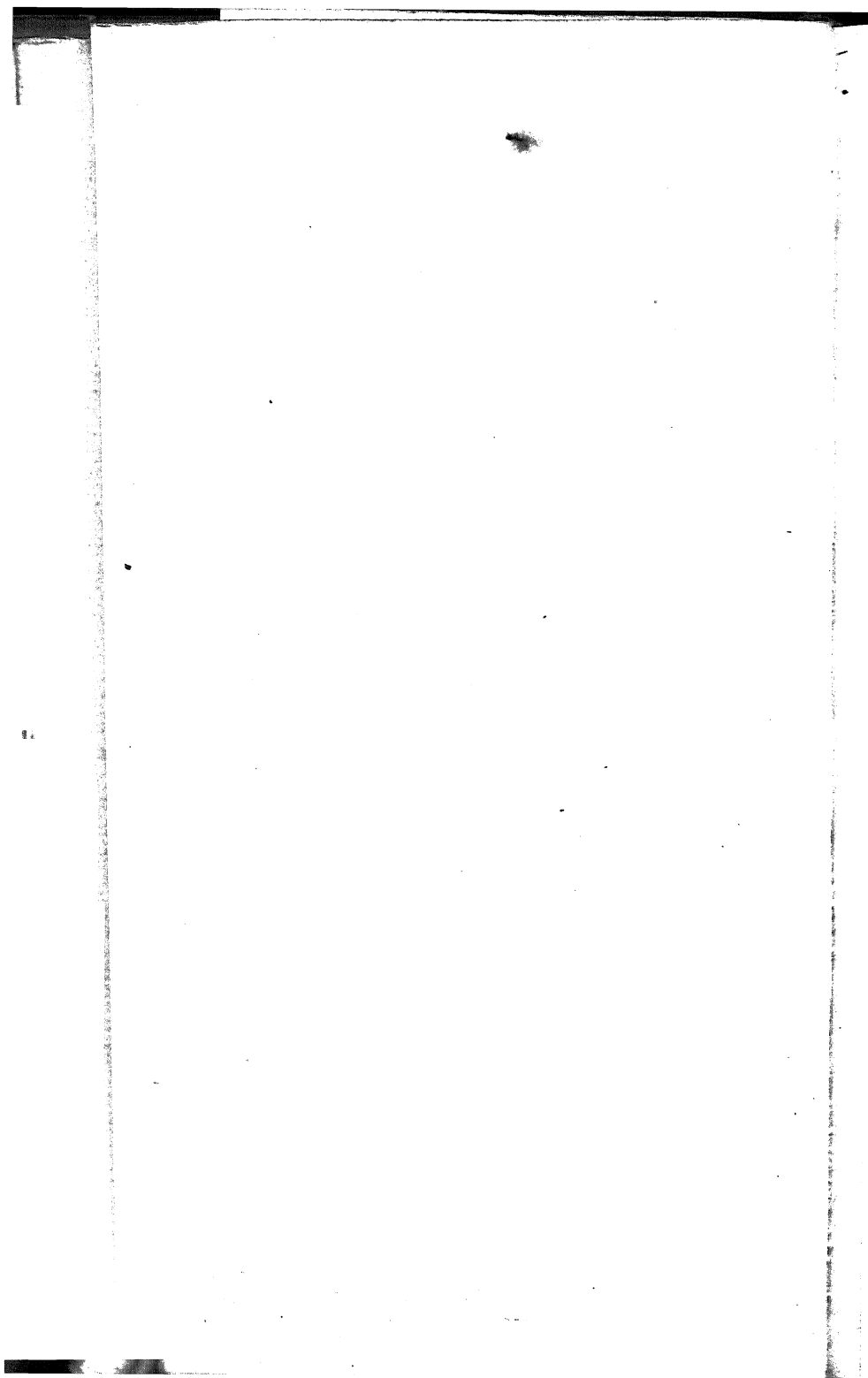


Fig. 25.





in the points A, C, and let the straight line D F, parallel to C F, cut the section or either of the opposite hyperbolas in D, F, and meet the tangent A F in G, and the straight line A C, joining the points of contact, in B; then the rectangle under D G, G F is equal to the square of G B.

For by Prop. XIII. $DG \times GF : AG^2 :: CF^2 : AF^2$. But as C F, B G are parallel, by Lemma V. (and 4. vi.) $CF^2 : AF^2 :: GB^2 : AG^2$, and therefore (II. v.) $DG \times GF : AG^2 :: GB^2 : AG^2$. Consequently (I4. v.) $DG \times GF = GB^2$.

Cor. I. The rest remaining as above, if from any point E in the tangent A F, there be drawn the straight line E H parallel to the tangent C F, and meeting A C in H, and if from the same point E there be drawn any straight line E I L, cutting the section or opposite hyperbolas in I and L; then the rectangle under I E, E L and the square of E H will be to one another as the squares of the tangents, or the rectangles under the segments of the secants meeting one another and parallel to I L, E H. For from the point G draw G N parallel to I L, and let it cut the curve or curves in M, and N. Then by Prop. XIII. $IE \times EL : MG \times GN :: AE^2 : AG^2$; and by similar triangles, and this Proposition, $AE^2 : AG^2 :: EH^2 : GB^2$ or its equal $DG \times GF$. Hence (II. v.) $IE \times EL : MG \times GN :: EH^2 : DG \times GF$, and, by alternation, $IE \times EL : EH^2 :: MG \times GN : DG \times GF$. Consequently, by Prop. XIII. (and II. v.) the Cor. is evident.

Cor. 2. If a straight line B H cut a conic section, or opposite hyperbolas in D, G, and meet in the points B, H two straight lines A B, C H which touch the section or opposite hyperbolas in A, and C, and if B H meet

A C



BOOK I.
 AC joining the points of contact in E; the rectangle under DB, BG will be to the rectangle under GH, HD as the square of BE to the square of HE. For if the tangents AB, CH be parallel, the triangles (29. i.) ABE, CHE will be equiangular, and therefore, by the fifth Lemnia, (and 4. vi.) in this case $AB^2 : CH^2 :: BE^2 : HE^2$; and by Prop. XIII. $AB^2 : CH^2 :: DB \times BG : GH \times HD$. In this case therefore (11. v.) the Cor. is evident. But if the tangents be not parallel, through H draw the straight line LK parallel to AB, and let it meet AC in K, and the curve or curves in the points L, M. Then the triangles ABK, KHE are (29. i.) equiangular, and as above $AB^2 : HK^2 :: BE^2 : HE^2$. But by this Proposition HK^2 is equal to $LH \times HM$, and therefore $AB^2 : LH \times HM :: BE^2 : HE^2$; and by Prop. XIII. $AB^2 : LH \times HM :: DB \times BG : GH \times HD$. Consequently (11. v.) $DB \times BG : GH \times HD :: BE^2 : HE^2$.

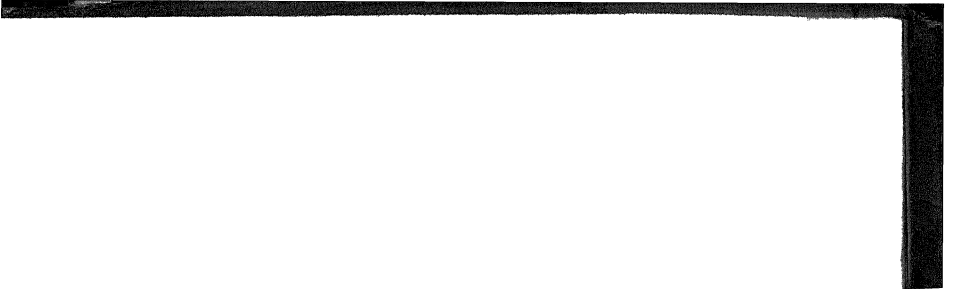


Fig. 31.

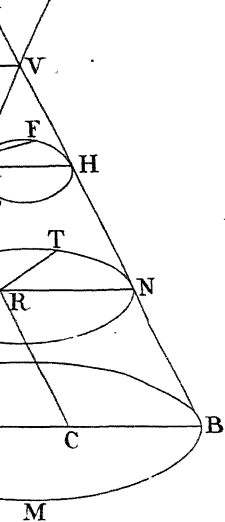


Fig. 30.

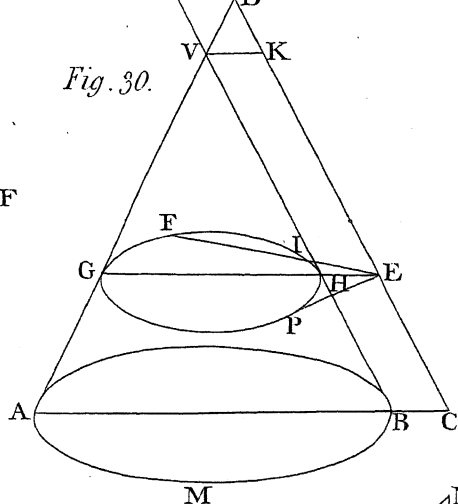


Fig. 32.

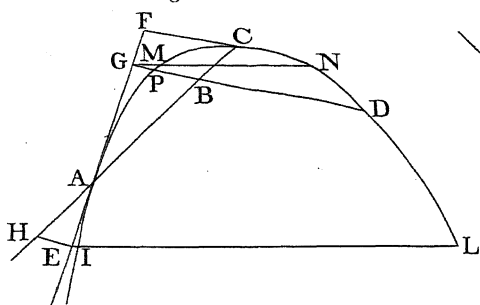


Fig. 33.

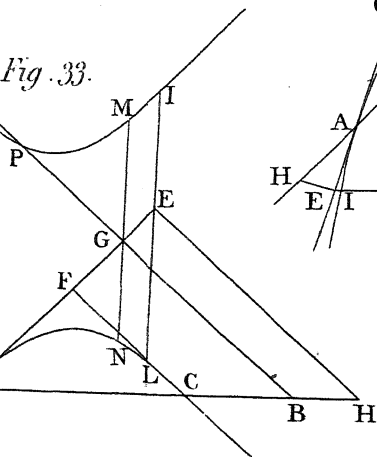
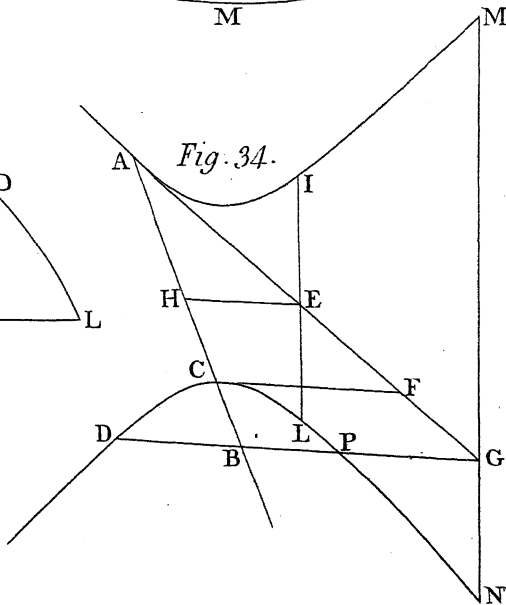
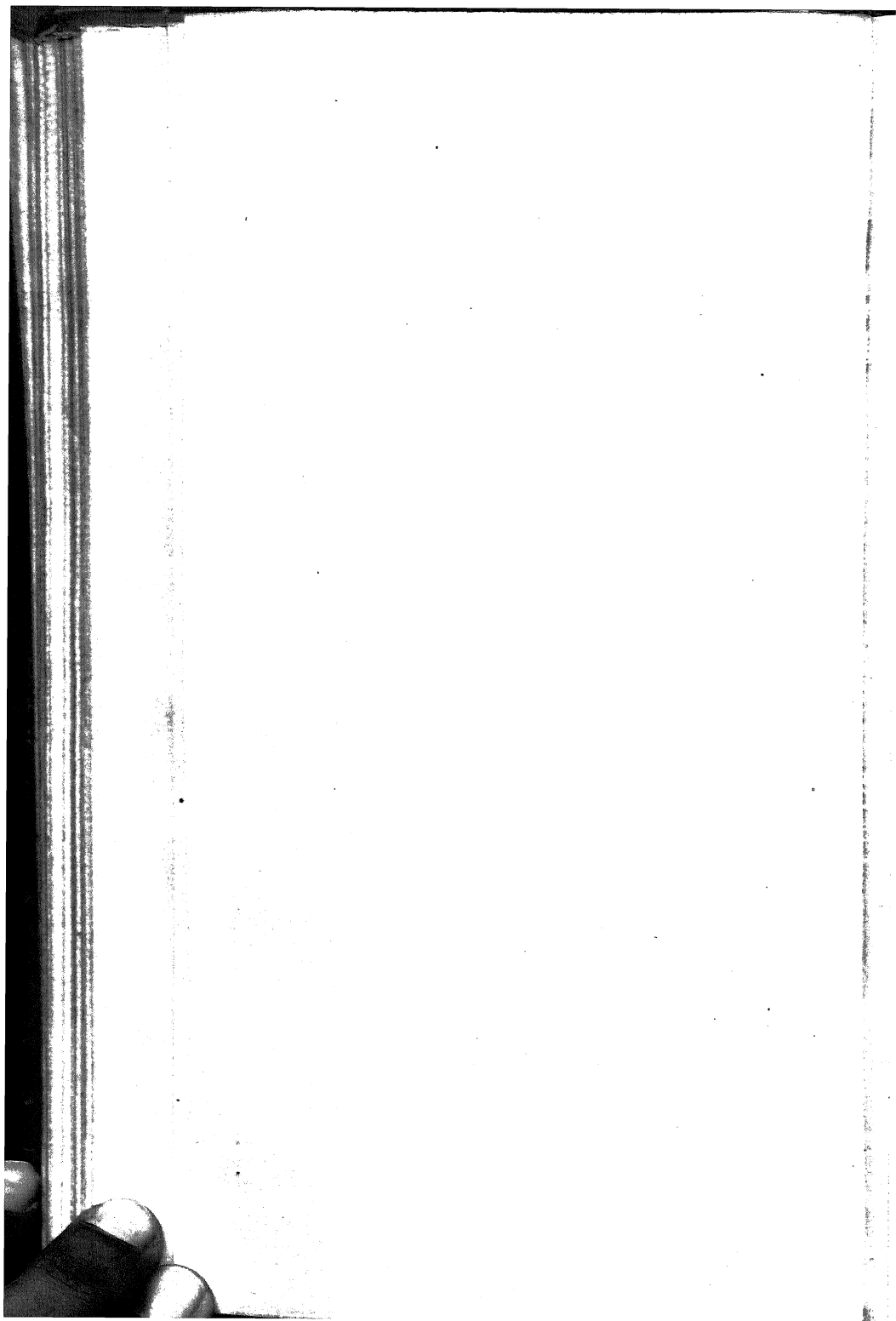


Fig. 34.





A
GEOMETRICAL TREATISE
OF
CONIC SECTIONS.

BOOK II.
Of the Ellipse and Hyperbola.

DEFINITIONS.

I.

THAT point within an ellipse or between opposite hyperbolas, in which every straight line, passing through it and terminated by the curve or opposite curves, is bisected, is called *the Center* of the ellipse, or *the Center* of the hyperbola or opposite hyperbolas.

II.

Any straight line passing through the center of an ellipse, and terminated by the curve, is called a *Diameter* of the ellipse.

III.

A straight line passing through the center of opposite hyperbolas, and terminated by the opposite curves, is called

BOOK II. called a *Transverse Diameter* of the opposite hyperbolas, or of either of the opposite hyperbolas. And a straight line passing through the center of opposite hyperbolas, and bisecting a straight line not passing through the center, and terminated by the opposite curves, is called a *Second Diameter* of the opposite hyperbolas, or of either of the opposite hyperbolas.

IV.

Any straight line not passing through the center of an ellipse, or opposite hyperbolas, terminated by the curve of the ellipse or either hyperbola, or by the opposite curves, and bisected by a diameter, is called a *Double Ordinate* to the bisecting diameter; and its half is simply called an *Ordinate* to it.

V.

The points in which any diameter of an ellipse meets the curve, or in which any transverse diameter of opposite hyperbolas meets the opposite curves, are called the *Vertices* of the diameter; and the segments of a diameter, between an ordinate and its vertices, are called *Abscissas*.

VI.

Two diameters of an ellipse, or opposite hyperbolas, of which each bisects all straight lines terminated by the curve, or opposite curves, and parallel to the other, are called *Conjugate Diameters*.

VII.

A diameter of an ellipse, or opposite hyperbolas, which cuts its ordinates at right angles is called an *Axis* of the ellipse, hyperbola, or opposite hyperbolas.

PROP. I.

If two parallel straight lines touch an ellipse, or opposite hyperbolas, the straight line joining the points of contact will

will bisect any straight line parallel to them, and terminated by the curve of the ellipse, or by the curve of either of the opposite hyperbolas.

BOOK
II.

Let the two parallel straight lines GH, LM touch the ellipse $ADBE$, or the opposite hyperbolas FBN, IAK , in the points B, A ; the straight line AB , joining the points of contact, will bisect any straight line FN , parallel to the tangents, and terminated by the curve of the ellipse, or by the curve of either of the opposite hyperbolas.

Fig. 37.
38.

Let FN meet the curve in the points F, N , and AB in O . Through the points F, N draw GL, HM parallel to AB . Let GL meet the curve of the ellipse again, or the curve of the opposite hyperbola in I , the tangent GH in G , and the tangent LM in L . Let HM meet the curve of the ellipse again, or the curve of the opposite hyperbola in K , the tangent GH in H , and the tangent LM in M . Then, by Prop. XIII. Book I. $BG^2 : IG \times GF :: AL^2 : FL \times LI$; and (34. i.) as BG is equal to AL , and therefore the square of BG equal to the square of AL , we have (14. v.) $IG \times GF$ equal to $FL \times LI$. Consequently, by the sixth Lemma, FG is equal to IL . For the same reasons NH is equal to KM , and (34. i.) as FG is equal to NH , IL is equal to KM ; and (34. i.) as LG is equal to MH , IG is equal to KH , and $IG \times GF$ is equal to $KH \times HN$. Again, by Prop. XIII. Book I. $IG \times GF : GB^2 :: KH \times HN : HB^2$; and therefore (14. v.) the square of GB is equal to the square of HB , and GB is equal to HB . Consequently (34. i.) FO is equal to ON .

If, in the ellipse, the straight line DE , meeting the curve in D, E , be parallel to the tangents GH, LM , and if the straight line PDT parallel to AB touch the ellipse in D , then the straight line REA parallel to AB will

BOOK will touch the ellipse in E, and DE will be bisected in
 II. c, the point in which it meets AB. For let P D T meet
 the tangent GH in T, and the tangent LM in P; and
 let REQ meet the tangent GH in Q, and the tangent
 LM in R. Then, by Prop. XIII. Book I. $BT^2 : DT^2 :: AP^2 : PD^2$; and (34. i.) as BT, AP are equal; the
 square of BT is therefore equal to the square of AP,
 and consequently (14. v.) the square of DT is equal to
 the square of DP, and DT is equal to DP. But (34. i.)
 EQ, DT are equal to one another, as are also ER, DP
 to one another; and therefore EQ, ER are equal to
 one another. Now if REQ could meet the curve in
 any other point besides E, as in V, then as BQ (34. i.)
 is equal to AR, it might be proved as above, by means
 of Prop. XIII. Book I. that $VQ \times QE$ is equal to $ER \times RV$. It would therefore follow, by the sixth Lem-
 ma, that EQ is equal to RV; which by the above is
 absurd. Consequently REQ touches the ellipse in
 E, and therefore, by the Cor. to Prop. XIII. Book I.
 $EQ : QB :: DT : TE$; and as EQ, DT are equal, we
 have (14. v.) BQ equal to BT. Consequently CD is
 equal to CE, for (34. i.) CE is equal to BQ, and CD
 to BT.

Cor. If two parallel straight lines, as GH, LM touch-
 ing an ellipse or opposite hyperbolas meet a straight
 line, as FI, which cuts the ellipse or opposite hyper-
 bolas, and is parallel to the straight line joining the
 points of contact, the segments of the secant between
 the curve or curves and the tangents will be equal to
 one another; for by the above FG is equal to IL. And
 if two parallel straight lines touching an ellipse meet a
 straight line which touches the ellipse, and is parallel
 to the straight line joining the points of contact, the
 segments of the last mentioned tangent, between the
 point of contact and the parallels, will be equal to one
 another.

her. This is also evident from the above, for it is proved that DP is equal to DT . BOOK
II.

PROP. II.

Two parallel straight lines touch an ellipse or opposite hyperbolas, and the straight line joining the points of contact be bisected, the point in which it is bisected will be the center of the ellipse or opposite hyperbolas; and no other point can be a center of the ellipse or opposite hyperbolas.

Let the two parallel straight lines GH , LM touch the ellipse $ADBE$, or opposite hyperbolas FBN , IAK the points B , A , and let the straight line AB , joining points of contact, be bisected in c ; the point c is the center of the ellipse or opposite hyperbolas, and no other point, besides c , can be the center of the ellipse or opposite hyperbolas. Fig. 37.
38.

Part I. Take any other point N in the curve of the ellipse, or in the curve of either of the opposite hyperbolas. Then Nc being drawn, and produced, it will meet the curve of the ellipse, or the curve of the opposite hyperbola, and the whole line terminated by the curve, or curves, will be bisected in c . For let Nc be parallel to GH , LM , and draw NOF parallel to AB , and let it meet AB in O , and the curve again in F .

Make As equal to BO , and through s draw ISK parallel to the tangents GH , LM , or to NOF , and let K be one of the points in which it meets the curve. Then, by Prop. I. NF will be bisected in O , and SK will be half the whole line, of which it is a part, terminated by the curve; and therefore, by Prop. XIII. Book I. $AO \times OB : NO^2 :: BS \times SA : KS^2$. But as BO are equal to one another, $AO \times OB$ is equal to $AO \times SA$, and therefore (14. v.) the square of NO is equal

BOOK
II.

equal to the square of ks , and no is equal to ks . Let nc produced meet ksi in i , and then as cs , co are equal, and (29. i.) the triangles $nc o$, $i c s$ equiangular, it follows (26. i.) that nc is equal to ci , and is is equal to no ; and therefore, by the above, is is equal to sk . Consequently, by Prop. I. the point i is in the curve, and therefore nc being produced, it meets the curve of the ellipse again, or the curve of the opposite hyperbola, and the whole line ni , terminated by the curve, or curves, is bisected in c . The point c is therefore the center of the ellipse or opposite hyperbolas; for in the ellipse the straight line parallel to the tangents gh , lm , and passing through c is also bisected in c , by Prop. I.

Part II. No other point, besides c , can be the center of the ellipse, or opposite hyperbolas. In the ellipse this is evident; for if there could be another, then a straight line passing through c and that other center, and terminated by the curve, would, by the second Definition, be bisected in two points: which is absurd.

Fig. 38.

Nor can the hyperbolas fbn , iak have any other center besides c . For through c draw db parallel to gh , lm ; and suppose the point d in this straight line to be another center. Through d draw the straight line fi parallel to ab , and let it meet the tangents in g , l , and the opposite curves in f , i . Then, by Cor. Prop. I. fg is equal to il ; and (34. i.) as gd is equal to bc , and dl to ca , and bc equal to ca , it follows that fd , di are equal. But through d draw a straight line at , not parallel to ab , and let it meet the curves in a , t ; which it evidently may do, by Prop. IX. Book I. as it may be drawn parallel to a straight line drawn from a point in the curve fbn through c and meeting the opposite curve, by Part I. Let at meet the straight lines fn , ik parallel to the tangents gh ,
 lm ,

L M, in **R** and **P**. Let **R** be between the points **F**, **N**, **BOOK**
 and then **P** will be without **I**, **K**, or without the hyper- **II.**
 bola **I A K**. Then, by the above, as **F D** is equal to
D I, and (29. i.) the triangles **F D R**, **I D P** equiangular,
 it follows (26. i.) that **R D** is equal to **P D**. Conse-
 quently **T Q** is not bisected in **D**, and therefore **D** is not
 a center. Nor can any point out of the line **D E** be a
 center; for if it could, then a straight line drawn
 through it, and parallel to **A B**, and meeting the curves,
 would, by the above, be bisected by **D E**, and, according
 to the first Definition, it would also be bisected in this
 other center. The same straight line would therefore
 be bisected in two points: which is absurd. Conse-
 quently no other point besides **c** can be a center.

Cor. 1. A straight line, as **A B**, joining the points of **Fig. 37.**
 contact of two parallel tangents **G H**, **L M**, is a dia- **38.**
 meter of the ellipse, or opposite hyperbolas; and **D E**
 drawn through the center **c**, and parallel to the tan-
 gents, or to the secants **F N**, **I K** parallel to them, is
 also a diameter. For in the ellipse **D E** is a diameter,
 according to the second Definition; and in the hyper-
 bola **D E** is a second diameter, by the third Definition,
 as it bisects, by the above, any straight line parallel to
A B, and terminated by the curves of the opposite hy-
 perbolas.

Cor. 2. If two straight lines **A L**, **B G** touch an el-
 lipse, or opposite hyperbolas in **A**, **B**, the vertices of the
 diameter **A B**, they will be parallel. For if **B G** be not
 parallel to **A L**, the tangent parallel to **A L** will meet
 the curve of the ellipse, or the curve of the opposite hy-
 perbola, not in **B**, but in some other point, as **N**, as two
 straight lines cannot touch a conic section in the same
 point. Then, by the preceding Cor. if **A N** be drawn,
 it will be a diameter: which is absurd. For the

BOOK II. straight line AN cannot pass through c , the center, as c is in the diameter AB .

PROP. III.

If two parallel straight lines touch an ellipse or opposite hyperbolas, straight lines parallel to them in the ellipse, or in either hyperbola, will be ordinates to the diameter joining the points of contact; but, in the opposite hyperbolas, straight lines parallel to the diameter joining the points of contact will be ordinates to the second diameter parallel to the tangents; and, in either case, ordinates to the same diameter of an ellipse, or opposite hyperbolas, are parallel to one another.

Fig. 37.
38.

Part I. Let two parallel straight lines GH , LM touch the ellipse $ADBE$, or opposite hyperbolas FBN , IAK in the points A , B , and let FN , IK , in the ellipse or in either hyperbola, be parallel to GH , LM ; then AB will be a diameter by Cor. I. Prop. II. and by Prop. I. it will bisect FN , IK . This part is therefore evident, by the above and the fourth Definition.

Fig. 38.

Part II. The rest remaining as above, let DE be a second diameter, parallel to the tangents GH , LM , and let FI , NK , terminated by the opposite curves, be parallel to AB , and then by Cor. I. Prop. II. FI , NK are bisected by DE , and are therefore ordinates to it, according to the fourth Definition.

Fig. 39.
40.

Part III. Ordinates to the same diameter of an ellipse, or opposite hyperbolas, are parallel to one another. For first let AB be any diameter of the ellipse, or any transverse diameter of the opposite hyperbolas; and c being the center, let LA , GB be the parallel tangents drawn through the vertices A , B , according to Cor. 2. Prop. II. Then, by Part I. any straight line parallel

BOOK
II.

parallel to LA , or GB in the ellipse, or in either of the opposite hyperbolas, will be an ordinate to AB . But, if it be possible, let the straight line IP in the ellipse, or in either of the opposite hyperbolas, be an ordinate to the diameter AB , and not be parallel to LA , GB . Draw IK parallel to LA or GB , and let it meet AB in s , and the curve in K . Draw IC , and, being produced, let it meet the curve of the ellipse again, or the curve of the opposite hyperbola, in N ; and draw KN . Then, by Part I. IK is bisected in s ; and, as IN is bisected in c , $IC : CN :: IS : SK$, and therefore (2. vi.) AB, KN are parallel. Consequently, if IP meet AB in R and KN in V , (2. vi.) $IS : SK :: IR : RV$, and therefore IV is bisected in R . But the straight line KN in Fig. 39. is wholly within the ellipse, and in Fig. 40. NK is without the opposite hyperbolas, and being produced it falls on the one side within one hyperbola, and on the other within the opposite hyperbola. The other point P therefore, in which IP meets the curve, cannot be in KN , and consequently IP cannot be bisected in R , or be an ordinate to AB .

Lastly, the rest remaining as above, let DE be a second diameter of the hyperbolas parallel to the tangents LA, GB . Then any straight line parallel to AB , and meeting the opposite curves will be bisected by DE , according to Part II. But, if it be possible, let IX meet the opposite curves in I, X , the diameter DE in r , and be an ordinate to DE , and not be parallel to AB . Let IF be parallel to AB and meet the opposite curves in I, F , and DE in D . Draw IC , and, being produced, let it meet the opposite curve in N . Draw NF , and let it meet IX in Z . Then, as IN is bisected in c , and as IF , according to Part II. is bisected in D , $IC : CN :: ID : DF$, and therefore (2. vi.) NF is parallel

Fig. 40.

BOOK II. rallel to DE. Consequently (2. vi.) $ID : DF :: IY : YZ$, and therefore IY is equal to YZ . The straight line IX therefore cannot be bisected in Y , or be an ordinate to DE. In every case therefore, ordinates to the same diameter are parallel to one another.

Cor. 1. If a straight line bisect two parallel straight lines in an ellipse or hyperbola, or opposite hyperbolas, it will be a diameter: and straight lines drawn through its vertices, and parallel to the lines bisected, will touch the ellipse, or opposite hyperbolas, if in the opposite hyperbolas it be a transverse diameter. For by *Cor. 2. Prop. VIII. Book I.* two straight lines, and only two, parallel to the lines bisected, can be drawn to touch the ellipse or opposite hyperbolas; and by *Cor. 1. Prop. II.* the straight line joining the points of contact is a diameter, and, by this Proposition, this diameter will bisect only such straight lines in the ellipse or opposite hyperbolas as are parallel to the tangents. It is also evident from this Proposition that a straight line bisecting two parallel straight lines, terminated by the curves of opposite hyperbolas, is a second diameter of the hyperbolas.

Cor. 2. If in an ellipse, hyperbola, or opposite hyperbolas, a diameter bisect a straight line not passing through the center, it will also bisect any line parallel to it in the same section or opposite hyperbolas. For, by the preceding *Cor.* a straight line bisecting two parallel straight lines in an ellipse, hyperbola, or opposite hyperbolas, is a diameter, and therefore passes through the center. Consequently a diameter, or a straight line passing through the center, and bisecting one of two parallel lines in the same section, or opposite hyperbolas, will also bisect the other.

Cor. 3. A diameter of an ellipse or hyperbola will bisect all straight lines in the section parallel to a tangent passing

passing through its vertex; and ordinates to a diameter and tangents passing through its vertices are parallel to one another, BOOK
II.

PROP. IV.

Two diameters of an ellipse, or opposite hyperbolas, are conjugate diameters, if one of them be parallel to the ordinates of the other.

Let AB, DE , be two diameters of the ellipse $ADBE$, or of the opposite hyperbolas MA, GBF , and let the diameter DE be parallel to GF a double ordinate to AB ; the diameter DE will be the conjugate diameter to AB . Fig. 41.
42.

For let c be the center, and FC being drawn, and produced, let it meet the curve of the ellipse, or the curve of the opposite hyperbola in M . Draw GM , and let it meet DE in P . Let AB meet GF in H , and then as GF is a double ordinate to the diameter AB , it is bisected in H ; and as the diameter FM is bisected in c , the center, $FC : CM :: FH : HG$, and therefore (2. vi.) MG is parallel to AB . Again, as DE, GF are parallel, (2. vi.) $MC : CF :: MP : PG$, and therefore MG is bisected in P . Consequently the diameters DE, AB are conjugate to one another, according to the sixth Definition; for by Cor. 2. Prop. III. DE will bisect any straight line parallel to MG or AB , and AB will bisect any straight line parallel to GF or DE , the straight lines parallel to MG or GF being terminated by the curve of the ellipse, hyperbola, or opposite hyperbolas.

Cor. 1. From the above, Cor. 3. Prop. III. and Cor. 1. Prop. II. it is evident, that if two parallel straight lines touch an ellipse or opposite hyperbolas, and if, from any point in the curve of the ellipse, or of either hyperbola, except the points of contact, a straight

BOOK
II.

line be drawn parallel to the tangents, it will be either an ordinate to the diameter joining the points of contact, or in the ellipse it will be the diameter conjugate to that joining the points of contact.

Cor. 2. Ordinates to a diameter, of an ellipse or opposite hyperbolas, tangents passing through its vertices, and its conjugate diameter are parallel to one another.

DEFINITIONS.

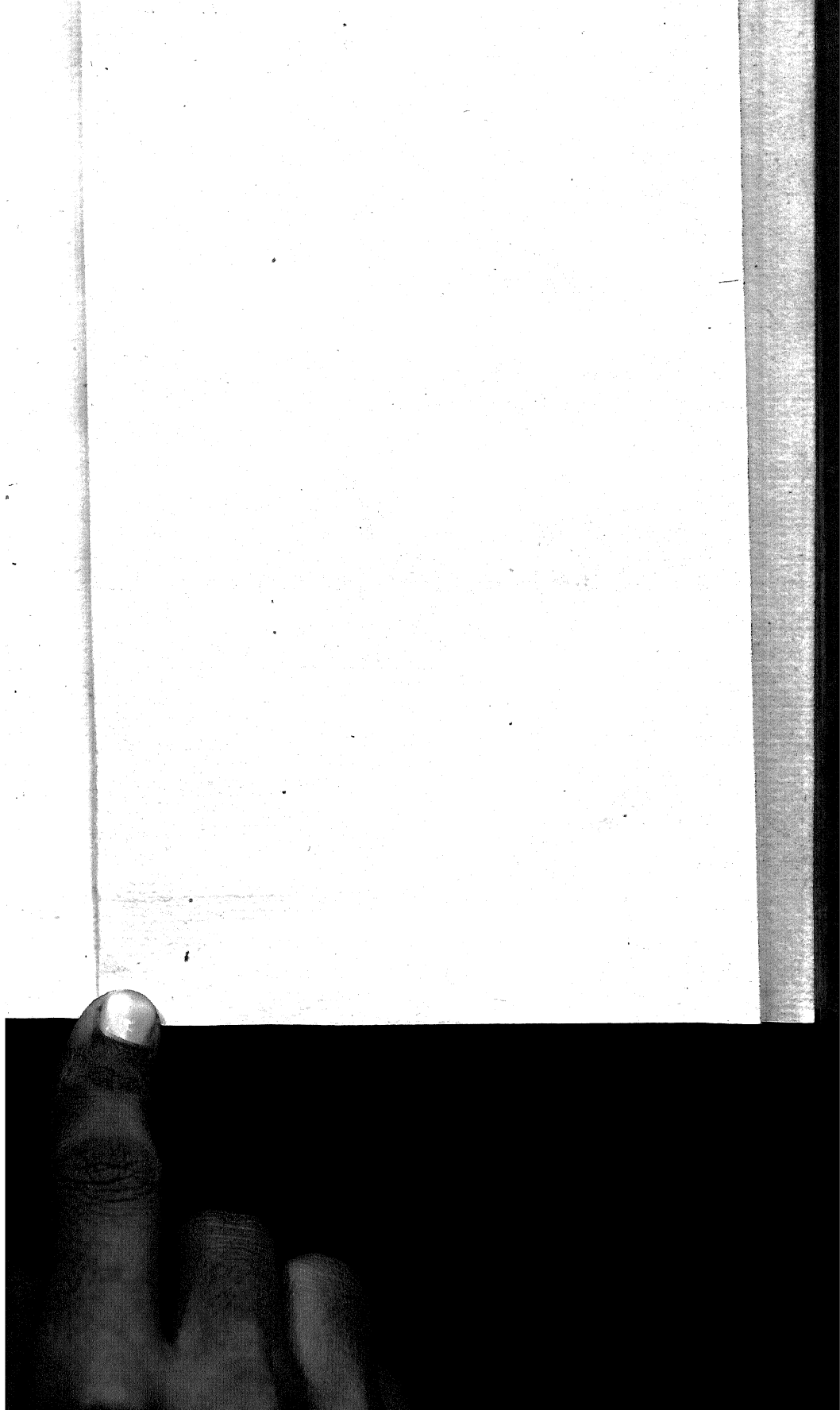
VIII.

Fig. 42. If c be the center of the opposite hyperbolas MA , FB , and AB a transverse diameter, to which DE is the conjugate, and HF an ordinate, and if the rectangle under AH , HB be to the square of HF as the square of CB to the square of CE or CD , the points D , E are called *the Vertices of the second Diameter DE*. In this way the magnitude of any second diameter is determined by its vertices.

Cor. If the ordinate HF be parallel to the base of the cone, in which the hyperbola was formed, the semi-diameter CE or CD will be equal to the straight line drawn through c and parallel to the base of the cone, and touching either of the conical superficies. For, in this case, by the second *Cor.* to *Prop. X.* Book I. the rectangle under AH , HB is to the square of HF as the square of CB to the square of the line drawn through c and parallel to the base of the cone, and touching either of the conical superficies; and, by this Definition, the square of CB is to the square of CE or DE in the same proportion. Consequently (9. v.) the *Cor.* is evident, as the sides of equal squares are equal.

IX.

A straight line which is a third proportional to two conjugate diameters of an ellipse, or opposite hyperbolas,



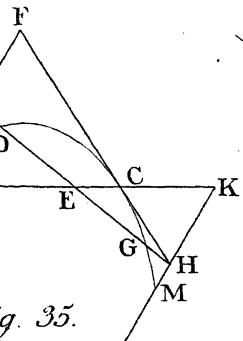


Fig. 35.

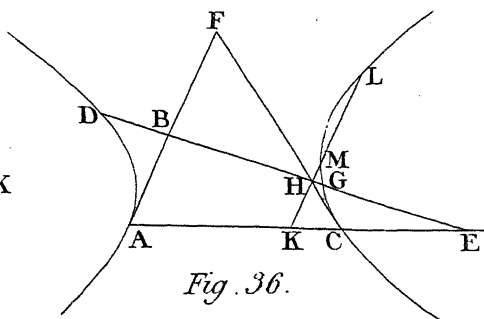


Fig. 36.

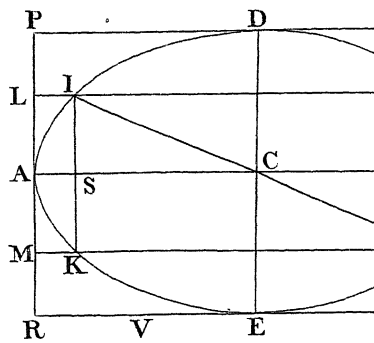


Fig. 37.

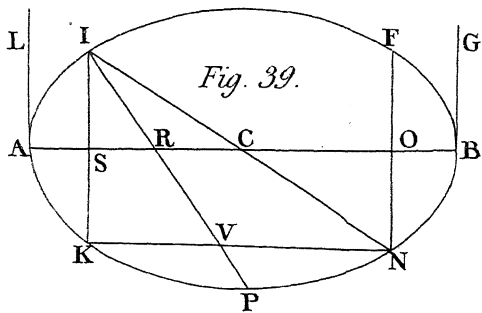


Fig. 39.

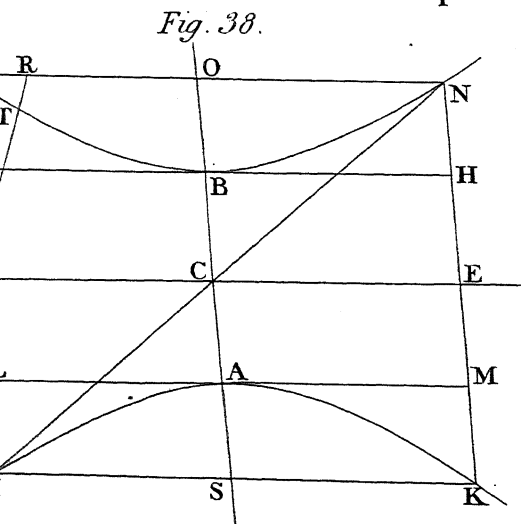


Fig. 38.

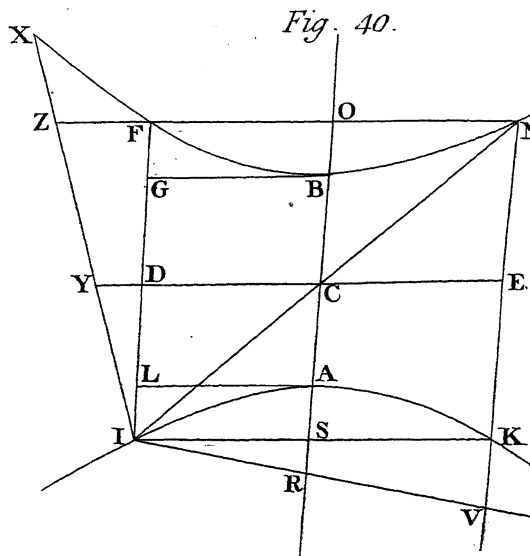
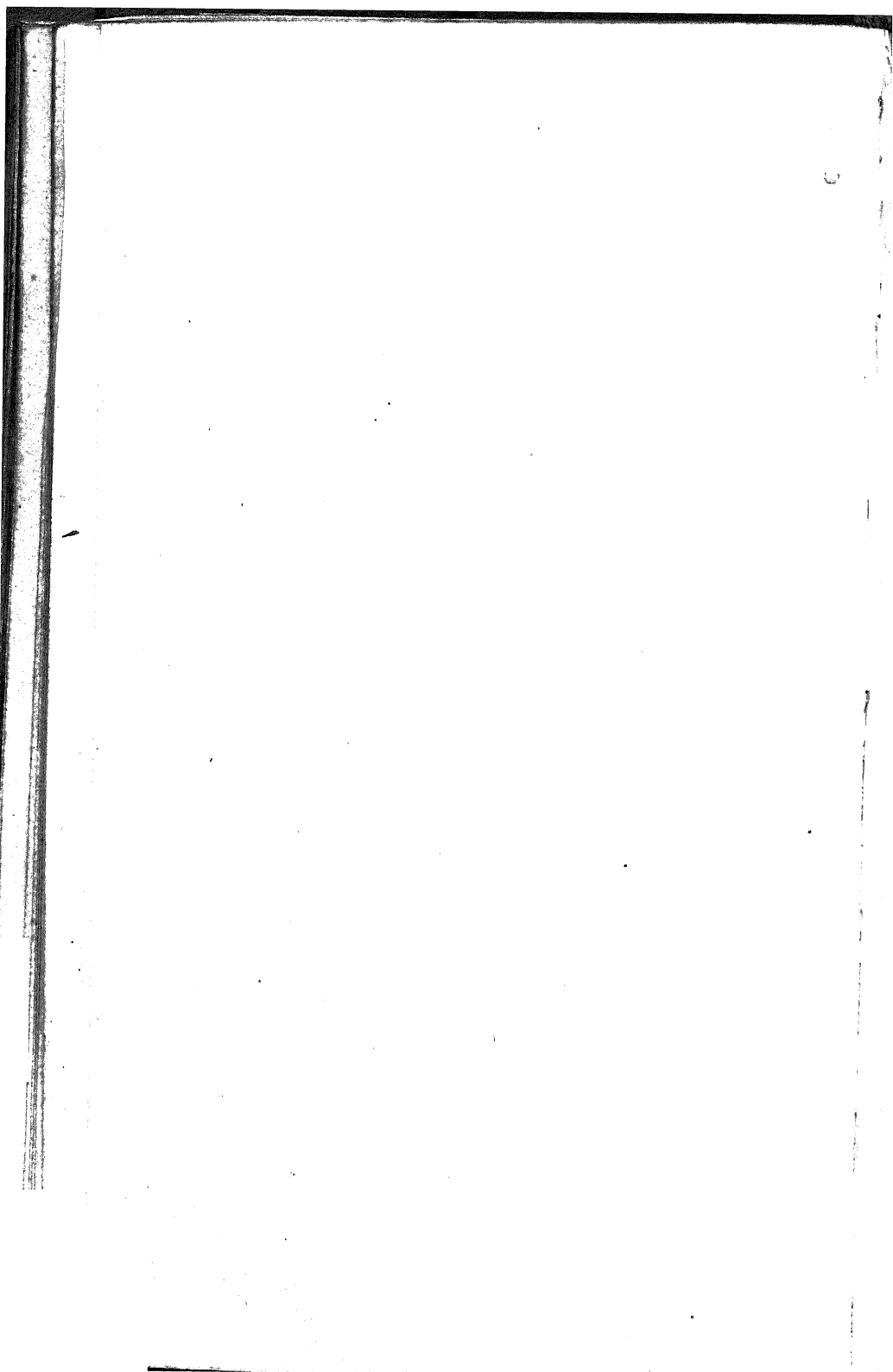


Fig. 40.



If each of two straight lines, meeting one another, touch or cut, or one of them touch, and the other cut, an ellipse, hyperbola, or opposite hyperbolas; the square of the first of the two, if a tangent, or the rectangle under its segments, if a secant, will be to the square of the second, if a tangent, or the rectangle under its segments, if a secant, as the square of the semidiameter parallel to the first to the square of the semidiameter parallel to the second.

In the ellipse, and when the two straight lines are parallel to two transverse diameters of opposite hyperbolas, the Proposition is evident from Prop. XIII. Book I. For diameters in the ellipse, and transverse diameters of opposite hyperbolas, are secants meeting one another in the center, in which they are bisected.

For other cases, first let LM , GH cutting either of the opposite hyperbolas in L , M and G , H , meet one another in K ; and let CK be the semidiameter parallel to LM , and let CF be the semidiameter parallel to GH , and let CK , LM be parallel to the base of the cone in which the hyperbola was formed; and then $LK \times KM : GK \times KH :: CE^2 : CF^2$. For, bisect GH in I , and through I draw NP parallel to LM or CK , and draw the transverse diameter IBA . Then, by Cor. 2. to Prop. X. Book I. and the Cor. to Definition VIII. $NI \times IP : AI \times IB :: CE^2 : CB^2$; and by Definition VIII. $AI \times IB : IH^2 :: CB^2 : CF^2$. Consequently,

$$NI \times IP : AI \times IB : IH^2$$

$$CE^2 : CB^2 : CF^2,$$

and (22. V.) $NI \times IP : IH^2 :: CE^2 : CF^2$. But, by Prop.

Fig. 43.



BOOK II. Prop. XIII. Book I. $NI \times IP : IH^2 :: LK \times KM : GK \times KH$; and therefore (II. v.) $LK \times KM : GK \times KH :: CE^2 : CF^2$.

Fig. 44. Secondly, let GH , LM cut one another in K , in the hyperbola as above, and let neither of them be parallel to the base of the cone in which the section was formed. Let GH be parallel to the semidiameter CF , and let LM be parallel to the semidiameter CE . Through K draw QR parallel to the base of the cone in which the section was formed, and let CS be the semidiameter parallel to QR , or to the base of the cone. Then, as above, we have the two following ranks of magnitudes proportionals,

$$LK \times KM : GK \times KR : GK \times KH \\ CE^2 : CS^2 : CF^2;$$

and therefore (22. v.) $LK \times KM : GK \times KH :: CE^2 : CF^2$.

Thirdly, the rest remaining as above, let the straight line VT meet one of the opposite hyperbolas in v , the other in t , and the straight line QR in K . Then, by Prop. XII. Book I. and Cor. to Definition VIII. yx being the transverse diameter parallel to VT , $VK \times KT : QK \times KR :: CX^2 : CS^2$.

Lastly, every thing remaining as in the two preceding cases, by the above we have the two following ranks of magnitudes proportionals,

$$VK \times KT : QK \times KR : GK \times KH \\ CX^2 : CS^2 : CF^2, \text{ and there-}$$

fore (22. v.) $VK \times KT : GK \times KH :: CX^2 : CF^2$.

In every case therefore, by the above, and Prop. XIII. Book I. (and II. v.) "If each of two straight lines," &c.

Cor. 1. If two straight lines be ordinates to any diameter of an ellipse, or transverse diameter of an hyperbola, the square of the first will be to the square of the second,

second, as the rectangle under the abscisses corresponding to the first to the rectangle under the abscisses corresponding to the second.

Cor. 2. From this Proposition (and 22. vi.) it is evident, that if two straight lines meeting one another touch an ellipse, hyperbola, or opposite hyperbolas, they will be to one another as the femidiameters to which they are parallel.

Cor. 3. From this Proposition, and the first Lemma, it is evident, that if two conjugate diameters of an ellipse cut one another at right angles, they cannot be equal to one another; for if they were equal to one another, the section would be a circle, by the first Lemma, as the square of the ordinate would be equal to the rectangle under the corresponding abscisses.

PROP. VI.

If a straight line be an ordinate to any diameter of an ellipse, or any transverse diameter of an hyperbola, the rectangle under the abscisses of the diameter will be to the square of the ordinate as the diameter to its parameter.

Let the straight line GF be an ordinate to the diameter AB of the ellipse BG , or to the transverse diameter AB of the hyperbola BG , and let BH be the parameter of AB ; the rectangle under the abscisses AF , FB is to the square of FG as AB to BH .

Fig. 45.
46.

Let C be the center of the ellipse or hyperbola, and let DE be the diameter parallel to GF , and consequently, by Prop. IV. the conjugate diameter to AB . Then, by the ninth Definition, $AB : DE :: DE : BH$; and therefore (Cor. 2. 20. vi.) $AB^2 : DE^2 :: AB : BH$. But (15. v.) $AB^2 : DE^2 :: CB^2 : CD^2$; and therefore (II. v.) $CB^2 : CD^2 :: AB : BH$. But, by Prop. V.

CB^2

BOOK CB² : CD² :: AF × FB : FG²; and consequently
 II. (II. V.) AF × FB : FG² :: AB : BH.

Cor. 1. Let the parameter BH be at right angles to the diameter AB, and from the other vertex A, draw AH. From the point F draw FK perpendicular to AB, and let it meet AH, or AH produced, in K. Complete the rectangle KB, and it will be equal to the square of the ordinate FG. For, as BH, FK are at right angles to AB, they are parallel, and therefore (4. vi.) AB : BH :: AF : FK; and consequently, by this Proposition (and II. V.) AF × FB : FG² :: AF : FK. But (I. vi.) AF : FK :: AF × FB : FK × FB, and therefore AF × FB : FG² :: AF × FB : FK × FB. Consequently (I4. V.) FK × FB is equal to FG².

Cor. 2. Complete the rectangle LABH, and let LH meet FK in M, and let KN, the side of the rectangle KB, opposite to BF, meet BH in N; then in the ellipse the square of the ordinate FG is less than the rectangle under the absciss FB and the parameter BH, by the rectangle MN, similar to LB, and having one of its sides equal to BF; but in the hyperbola, the square of the ordinate FG, is greater than the rectangle under the absciss BF and the parameter BH, by the rectangle MN similar to LB, and having one of its sides equal to BF. This is evident from the preceding Cor.

SCHOLIUM.

On account of the deficiency of the square of FG from the rectangle under FB, BH in Fig. 45. Apollonius called the section an ellipse; and on account of the excess of the square of FG above the rectangle under FB, BH in Fig. 46. he called the section an hyperbola.

From the properties demonstrated above these sections

Fig. 45. and 46.) = a , its parameter $BH = p$, the absciss $FB = x$, and the ordinate $FG = y$. Then $AF = a \mp x$, the negative sign applying to the ellipse, and the positive sign to the hyperbola. And by the similar triangles ABH , AFK , $a : p :: a \mp x : \frac{ap \mp px}{a} = FK$.

Consequently by the first Cor. to Prop. VI. $\frac{ap \mp px}{a} \times x = px \mp \frac{px^2}{a} = px \mp \frac{p}{a}x^2 = y^2$.

PROP. VII.

If a straight line, touching an ellipse or hyperbola, meet a diameter, and from the point of contact there be drawn an ordinate to the diameter; the semidiameter will be a mean proportional between the segments of the diameter, between the center and ordinate, and between the center and tangent.

First, let the straight line EM , touching the ellipse or hyperbola EIG in the point E , meet any diameter AI in the ellipse or transverse diameter of the hyperbola in M , and let EF be an ordinate to the diameter, and meet it in F , and let C be the center; the semidiameter CI is a mean proportional between the segments CF , CM .

For, let AB , ID be tangents passing through the vertices A , I , and meeting the tangent EM in B and D . Then, by Cor. to Prop. XIII. Book I. $EB : ED :: AB : ID$; and as, by Cor. 3. Prop. III. the straight lines AB , EF , ID are parallel, it is evident (from 10. vi.) that $EB : ED :: AF : IF$. And as (29. i.) the triangles BAM , DIM are equiangular, $AB : ID ::$

AM

Fig. 47-
48.



BOOK II. $AM : IM$. Hence (II. v.) $AM : IM :: AF : IF$, and (18. v.) $AM + IM : IM :: AF + IF : IF$; and by halving the antecedents it will be in the ellipse $CM : IM :: CI : IF$, but in the hyperbola $CI : IM :: CF : IF$. Consequently, by conversion, it will be in the ellipse $CM : CI :: CI : CF$; but in the hyperbola $CI : CM :: CF : CI$, and therefore $CM : CI :: CI : CF$.

Fig. 49.

Let AI be now a second diameter of the opposite hyperbolas GK , EL and EF an ordinate to it; and let the tangent EM meet it in M , and the transverse diameter KL , parallel to EF , in N . Then, by Prop. IV. KL , AI are conjugate diameters; and therefore EF being drawn parallel to AI , and meeting KL in P , it will be an ordinate to KL . Let c be the center, and then, by the above, $CP : CL :: CL : CN$, and therefore (Cor. 2. 20. vi.) $CP^2 : CL^2 :: CP : CN$. But (34. i.) CP , PF are equal, and EP is equal to FC , and (4. vi.) $EF : CN :: MF : MC$, and therefore $CP^2 : CL^2 :: MF : MC$. Hence (17. v. and 6. ii.) $KP \times PL : CL^2 :: CF : MC$; and (1. vi. and 11. v.) $KP \times PL : CL^2 :: CF^2 : CF \times MC$; and (16. v.) $KP \times PL : CF^2$ or $EP^2 :: CL^2 : CF \times MC$. But, by Def. viii. $KP \times PL : EP^2 :: CL^2 : CI^2$, and therefore (11. and 9. v.) $CF \times MC$ is equal to CI^2 , and consequently $CM : CI :: CI : CF$.

Fig. 47.
48.

Cor. 1. From the above (and 17. vi.) $CM \times CF$ is equal to CI^2 , and in the ellipse these equals being taken from CM^2 , we have (6. ii. and 2. ii.) $AM \times MI$ equal to $CM \times MF$. But, when AI is a transverse diameter in the hyperbola, CM^2 being taken from the equals, $CM \times CF$, CI^2 , we have (5. ii. and 3. ii.) $AM \times MI$ equal to $CM \times MF$.

Cor. 2. As $CM \times CF$ is equal to CI^2 , by taking from each CF^2 in the ellipse, we have (3. and 5. ii.)

CF

$c F \times F M$ equal to $A F \times F I$. But $A I$ being a transverse diameter in the hyperbola, by taking the equals $c M \times c F$, $c I^2$ from $c F^2$, we have (2. and 6. ii.) $c F \times F M$ equal to $A F \times F I$.

Cor. 3. When $A I$ is a diameter of the ellipse or transverse diameter of the hyperbola, by the demonstration of the first part of the Proposition, $A M : I M :: A F : F I$.

PROP. VIII.

If two straight lines, touching an ellipse, hyperbola, or opposite hyperbolas, meet one another, the diameter bisecting the line joining the points of contact will pass through the point of concurrence.

Let the two straight lines $E M$, $G M$ touch the ellipse or hyperbola $E I G$, or the opposite hyperbolas $E L$, $G K$, in the points E , G , and meet one another in M , and let the diameter $c F$ bisect $E G$, the straight line joining the points of contact in F ; the diameter $c F$ will pass through M .

Fig. 47.
48.
49.

For let c be the center, and let A , I be the vertices of the diameter; and then, as $E G$ is bisected by the diameter $c F$, $E F$ is an ordinate to it, and therefore, by Prop. VII. $c F : c I :: c I$: the segment of the diameter intercepted between c and the tangent $E M$. For the same reasons $c F : c I :: c I$: the segment of the diameter intercepted between c and the tangent $G M$. Consequently the segment of the diameter between c and the tangent $E M$, is equal to the segment of the diameter between c and the tangent $G M$. The diameter must therefore pass through M ; for if it did not, it would, upon being produced, first meet the one tangent, and then the other, and its segments between c and the tangents would be unequal.

Cor.

ing the line joining the points of contact, will be a diameter.

PROP. IX.

If two parallel straight lines touching an ellipse, or opposite hyperbolas, meet a third tangent, the rectangle under their segments, between the points of contact and the points of concurrence, will be equal to the square of the semidiameter to which they are parallel; and the rectangle under the segments of the third tangent, between its point of contact and the parallel tangents, will be equal to the square of the semidiameter to which it is parallel.

Fig. 50.
51.
52.

Let the parallel straight lines AB , ID touch an ellipse ER , or opposite hyperbolas A, I , in the points A, I , and let them meet in B, D a straight line BD , which touches the ellipse, or one of the opposite hyperbolas in E ; then the rectangle under the segments AB , ID is equal to the square of the semidiameter parallel to AB or ID , and the rectangle under the segments BE , ED is equal to the square of the semidiameter parallel to BD .

For let c be the center, and draw EG parallel to AB , ID ; and draw also AI . Then by Cor. 1. to Prop. II. AI is a diameter, and, by Cor. 1. to Prop. IV. EG is either an ordinate to AI , or in the ellipse the conjugate diameter to it.

Fig. 50.

First, let EG be the conjugate diameter to AI , and then by Prop. IV. and Cor. 3. Prop. III. BD , AI are parallel to one another, and also AB , CE , ID to one another. Consequently (34. i.) AB , CE , ID are equal to one another, as are also AC , CI , BE , ED to one another;



Fig. 41.

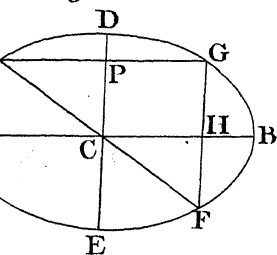


Fig. 42.

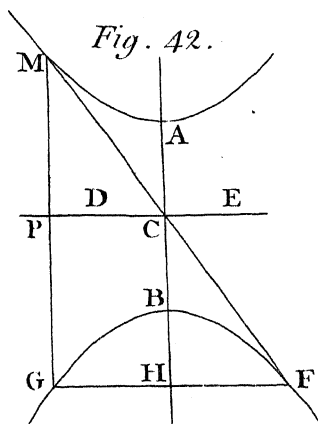


Fig. 43.

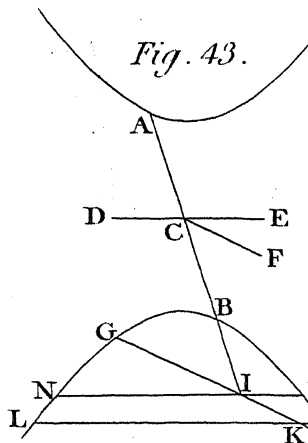


Fig. 44.

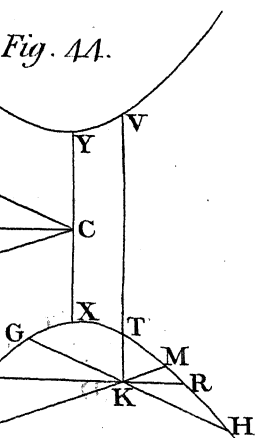


Fig. 45.

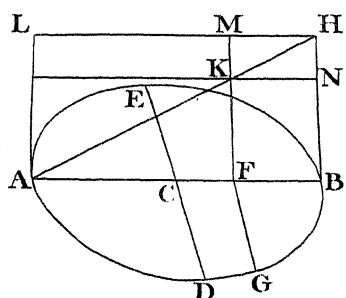


Fig. 46.

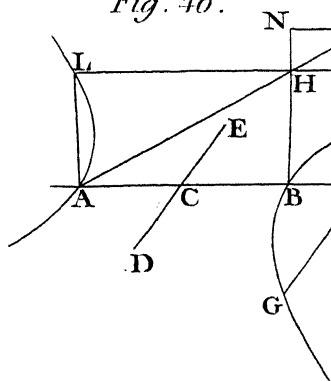


Fig. 47.

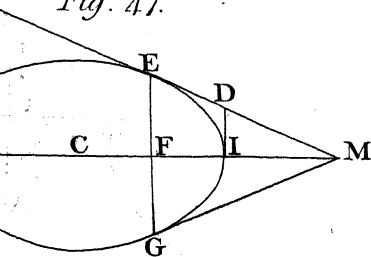


Fig. 48.

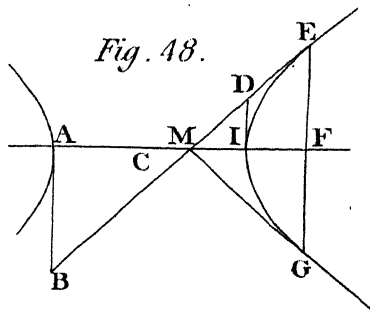
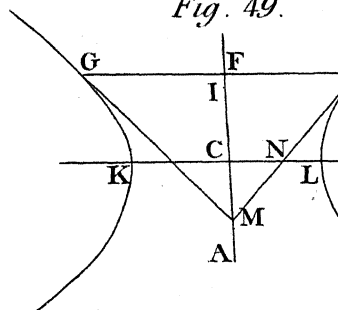
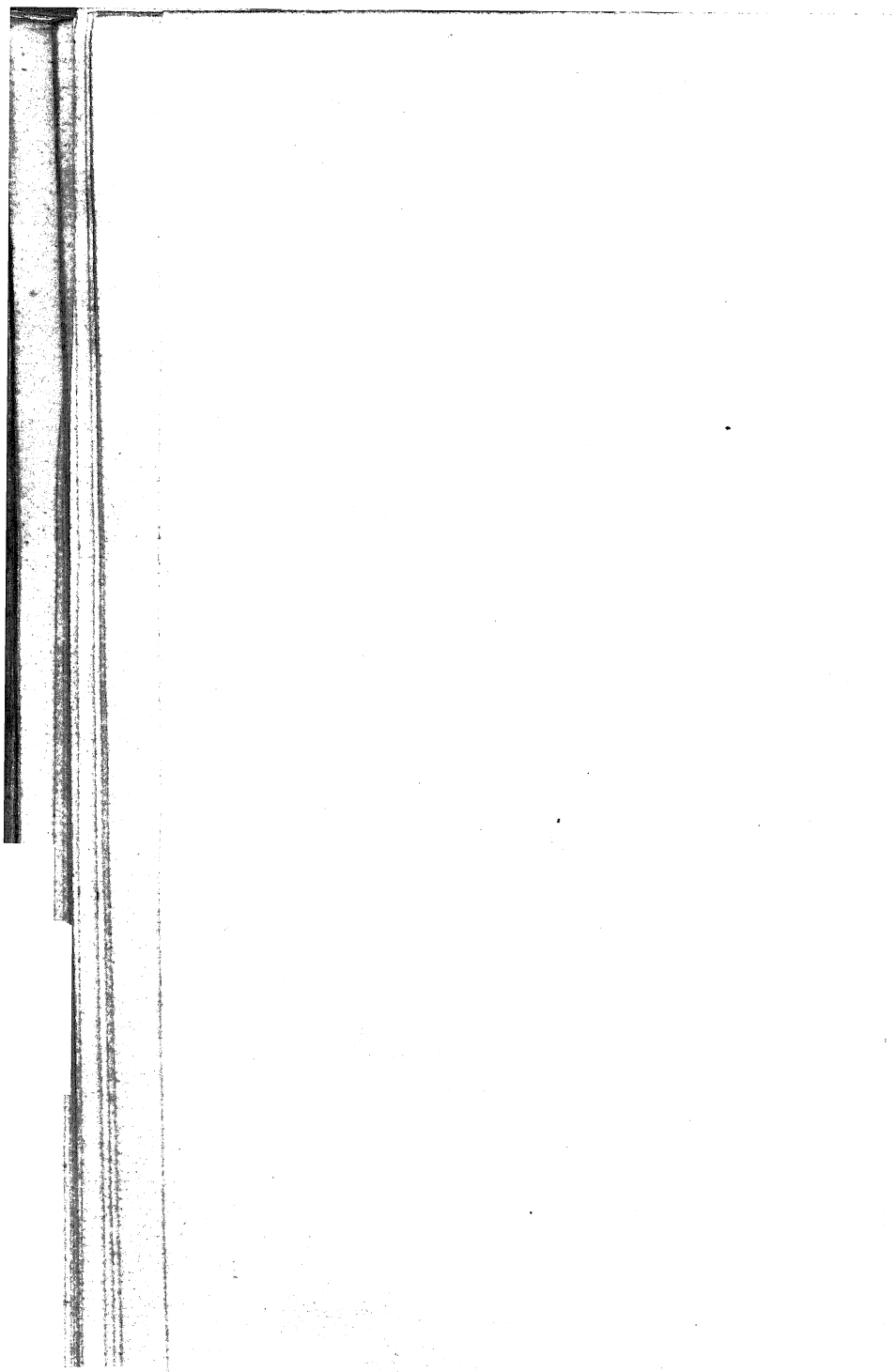


Fig. 49.





another; and therefore $AB \times ID$ is equal to CE^2 , and $BE \times ED$ is equal to AC^2 . BOOK II.

Next, let EG be an ordinate to the diameter AI , and let it meet it in F . Let CK be the semidiameter parallel to the tangents AB , ID , and CP the semidiameter parallel to the tangent EB ; and let EB meet the diameter AI in M , and the diameter KCL in H . Let EO be an ordinate to KCL , and let it meet it in O . Then, by Cor. 3. Prop. III. and Prop. IV. AB , LH , GE , ID are parallel, and EO is parallel to AI . By Cor. 1. Prop. VII. $AM \times MI$ is equal to $CM \times MF$, and therefore (16. vi.) $AM : CM :: MF : MI$; and (4. vi.) $AM : CM :: AB : CH$, and $MF : MI :: FE$ or $CO : ID$. Consequently (11. v.) $AB : CH :: CO : ID$, and (16. vi.) $AB \times ID$ is equal to $CH \times CO$, which is equal to CK^2 , by Prop. VII. (and 17. vi.) and therefore $AB \times ID$ is equal to CK^2 . Fig. 51.
52.

Again, by Cor. 2. Prop. V. $AB : BE :: CK : CP$, and by Cor. to Prop. XIII. Book I. $AB : ID :: BE : ED$. Hence (22. vi.) $AB \times ID : BE \times ED :: CK^2 : CP^2$, and as by the above $AB \times ID$ is equal to CK^2 , it follows (14. v.) that $BE \times ED$ is equal to CP^2 .

Cor. 1. Every thing remaining as above, as, by Cor. 2. Prop. VII. $AF \times FI$ is equal to $CF \times FM$, (16. vi.) $AF : FM :: CF : FI$. But on account of the parallels (10. vi.) $AF : FM :: BE : EM$; and $CF : FI :: HE : ED$. Consequently (11. v.) $BE : EM :: HE : ED$, and therefore (16. vi.) $ME \times EH$ is equal to $BE \times ED$ or to CP^2 .

Cor. 2. If a straight line as BE , touching an ellipse or hyperbola in the point E , meet the diameter AI in M , and the diameter KL in H , and if the rectangle under ME , EH be equal to the square of the semidiameter parallel to BE , the diameters AI , KL will be conjugate to one another.

BOOK
II.

PROP. X.

To find the axes of a given ellipse or hyperbola, the center being also given; and to demonstrate that the same section can have only two axes.

Fig. 53.
54.

Part I. Let $D B E$ be a given ellipse or hyperbola, of which c the center is also given; it is required to find the axes of the section.

Fig. 53.

In the ellipse draw $c G$, $c F$ two semidiameters, and, if they be unequal, let $c G$ be greater than $c F$. With c as a center, and a distance less than $c G$ but greater than $c F$, describe the circle $H D E$. Then from this construction and the nature of the two curves it is evident, that the circumference of the circle will cut the curve of the ellipse in four points, two of them being towards the left of the center, as the figure is viewed, and two of them towards the right. Let the circumference of the circle cut the curve of the ellipse in the points D , E . Draw the straight line $D E$; and through c draw $A B$ bisecting $D E$ in I , and (3. iii.) $A B$ will be at right angles to $D E$. Through c draw $L M$ parallel to $D E$, and $A B$, $L M$ will be the axes of the ellipse.

For, by construction, $D E$ is an ordinate to the diameter $A B$, and at right angles to it. Again, as $L M$ is parallel to $D E$, and as $A I D$ is a right angle, the angle $A C L$ (29. i.) is also a right one. By Prop. IV. the diameter $L M$ will also bisect all straight lines in the ellipse parallel to $A B$, and it will therefore cut its ordinates at right angles. Consequently $A B$, $L M$ are axes of the ellipse, according to the seventh Definition.

If the semidiameters $c G$, $c F$ be equal, then a diameter bisecting the angle $G c F$ will be one of the axes, and a diameter at right angles to it will be the other. For, in this case, if $G F$ be drawn, it will be bisected at right

right angles (4. i.) by the diameter bisecting the angle GCF ; and the rest will be as above. BOOK II.

Next, let the section DBE be an hyperbola, of which c is the center, and let K be any point within the hyperbola. With c as a center, and cK as a distance, describe the circle EKD , and let its circumference cut the curve of the hyperbola in the points E , D . Draw DE and bisect it in I , and through I draw the diameter AB ; and parallel to DE draw the diameter LM . The diameters AB , LM are the axes of the hyperbola. For DE is a double ordinate to the diameter AB , and (3. iii.) AB cuts it at right angles, and LM is parallel to the ordinate DE .

Fig. 54.

Part II. To demonstrate that an ellipse or hyperbola can have only two axes. First, let the section BDA be an ellipse, and c being the center, let RL , FG be the axes, found as above; and, if it be possible, let the diameter AB be also an axis. Let LD be a double ordinate to AB , meeting it in E , and the curve again in D ; and the diameter DK being drawn, let DH be an ordinate to RL *. Then, as by hypothesis AB is an axis, CEL , CED are right angles, and as LD is bisected in E , (4. i.) CL is equal to CD . Again, as by the above RL , FG are conjugate, DH is parallel to FG , by Prop. IV. and by Prop. V. $\text{CL}^2 : \text{CF}^2 :: \text{RH} \times \text{HL} : \text{DH}^2$. But, by Cor. 3. Prop. V. CL , CF must be unequal, and therefore, supposing CL to be the greatest, CL^2 is greater than CF^2 , and $\text{RH} \times \text{HL}$ greater than DH^2 . To these unequals add CH^2 , and then (5. ii. and 47. i.) CL^2 is greater than CD^2 ; and consequently CL is greater than CD . But CD is also equal to CL : which

Fig. 55.

* A method of drawing an ordinate to a given diameter will be inserted hereafter. The insertion of it previous to the above, or at this place, would have caused a needless repetition.

BOOK II. is absurd. The diameter AB therefore cannot be an axis.

Fig. 56.

Next, let the section EBD be an hyperbola, of which c is the center, FA the opposite hyperbola, and AB, LM the axes found as above; and, if it be possible, let the transverse diameter DF be an axis. Let DT touch the hyperbola in D , and meet the axis AB in T ; and let DI be an ordinate to AB , and let it meet it in I . Then in the triangle CDI , CDI is a right angle, by Def. VII. and therefore CDI is less than a right angle, and consequently CDT is much less than a right angle. But, by Cor. 2. Prop. IV. the tangent TD is parallel to the ordinates of the axis FD , and therefore, by Def. VII. (and 29. i.) the angle TDI is a right one. And, by the above, it is also less than a right one: which is absurd. Consequently DF is not an axis. Nor can a second diameter as GH , besides LM , be an axis. For FD conjugate to GH being drawn, and DT a tangent, the demonstration would end in the same absurdity.

Cor. From the above, and Prop. IV. it is evident, that the axes of an ellipse, or hyperbola, are conjugate diameters.

PROP. XI.

Of all the diameters of an ellipse the greater axis is the greatest and the lesser axis is the least; and of opposite hyperbolas the axes are the least diameters.

Fig. 55.

Part I. Let ABD be an ellipse, of which c is the center, RL the greater axis, and FG the lesser axis; of all the diameters RL is the greatest and FG the least.

For let KD be any other diameter, and let DH be an ordinate to RL , and DM an ordinate to FG . Then, by Cor. 2. Prop. IV. DH is parallel to FG , and DM to

RL ;

RL ; and therefore, by Prop. V. $CL^2 : CF^2 :: RH \times HL : DH^2$, and as CL is greater than CF , CL^2 is greater than CF^2 , and $RH \times HL$ is greater than DH^2 . To these add CH^2 , and (5. ii. and 47. i.) then CL^2 is greater than CD^2 . Consequently CL is greater than CD , and therefore RL is greater than KD . Again, by Prop. V. $CL^2 : CF^2 :: DM^2 : GM \times MF$, and therefore, as above, DM^2 is greater than $GM \times MF$. To these add the square of CM , and then (5. ii. and 47. i.) CD^2 is greater than CF^2 . Consequently KD is greater than FG .

Part II. Let AB , LM be the axes of the opposite hyperbolas EBD , AF , and FD , GH any other conjugate diameters; then AB is less than the transverse diameter FD , and LM is less than GH . Fig. 56.

For let DE be an ordinate to the axis AB , and let it meet it in I , and let DT touch the hyperbola in D , and let it meet AB in T . Let BP touch the hyperbola in the vertex B , and let it meet the tangent DT in P , and let C be the center. Then, by Def. VII. CDI is a right angle, and therefore (19. i.) CD is greater than CI , and consequently much greater than CB . Hence the transverse axis is less than the transverse diameter FD . Again, by Cor. 2. Prop. IV. GH is parallel to TD , and DI to PB ; and, by Prop. VII. $CI : CB :: CB : CT$, and therefore by conversion, $CI : BI :: CB : BT$. Hence, as CI is greater than CB , (14. v.) BI is greater than BT . But (2. vi.) $BI : BT :: DP : PT$, and therefore DP is greater than PT ; and, as (29. i.) PBT is a right angle, PT (19. i.) is greater than BP . Hence DP is greater than BP ; and as, by Cor. 2. Prop. V. $DP : BP :: CH : CL$, CH is greater (14. v.) than CL . Consequently the axis LM is less than the second diameter GH .

BOOK
II.

DEFINITIONS.

X.

Fig. 60.
61.

In the ellipse the greater axis is called the *Transverse Axis*, and the other axis is called the *Conjugate Axis*; and in the hyperbola the axis which is a transverse diameter is called the *Transverse Axis*, and the other axis is called the *Conjugate Axis*.

XI.

If c be the center, AB the transverse axis, and DE the conjugate axis of the ellipse ADB , or of the opposite hyperbolas AI , BF , then if in AB two points F , O be so taken that the rectangle under AF , FB , and also the rectangle under AO , OB , be equal to the square of CD or CE , the semiconjugate axis; the points F , O are called the *Foci*, or *Umbilici*, of the ellipse, hyperbola, or opposite hyperbolas.

Cor. 1. As (axiom i. 1.) $AF \times FB$ is equal to $AO \times OB$, the foci F , O are equally distant from the vertices A , B , by the sixth Lemma. It is also evident, that the foci are equally distant from the center.

Fig. 60.

Cor. 2. In the ellipse the distance of each of the foci from either extremity of the conjugate axis is equal to the semitransverse axis. For, supposing a straight line to be drawn from D to O , the square of DO (47. i.) will be equal to the squares of CO , CD together; and therefore, by this Definition, (and 5. ii.) the square of DO is equal to the square of AC . Consequently DO is equal to AC ; and therefore if with D or E as a center, and AC or CB as a distance, a circle be described, the circumference will cut AB in O , and F the foci.

Fig. 61.

Cor. 3. In the hyperbola the distance of each of the foci from the center is equal to the distance between the vertices of the transverse and conjugate axes. For, supposing a straight line to be drawn from D to A , the square of DA (47. i.) will be equal to the squares of

 CA ,

CA, CD together; and therefore, by this Definition, BOOK
II.
(and 6. ii.) the square of DA is equal to the square of CO or CF . Consequently CO or CF is equal to DA ; and therefore the foci F, O may be easily found from the axes.

Cor. 4. The double ordinate TS to the axis AB , Fig. 6c.
61.
drawn through either focus, suppose F , is equal to the parameter of the axis AB . For, by Prop. V. $CB^2 : CD^2 :: AF \times FB$, or by this Def. as $CD^2 : TF^2$; and therefore (22. vi.) $CB : CD :: CD : TF$. Consequently (15. 5.) $AB : DE :: DE : TS$, and the Cor. is evident from Def. IX.

XII.

If through the foci F, O of an ellipse ADB , or of the opposite hyperbolas AI, BP , ordinates FT, OI to the axis AB be drawn, and if through the points T, I in which they meet the curve straight lines HT, IL be drawn to touch the section, or opposite hyperbolas, the tangents HT, IL are called *Focal Tangents*.

PROP. XII.

If a tangent, passing through a vertex of the transverse axis of an ellipse or opposite hyperbolas, meet a focal tangent, its segment between the point of contact and point of concurrence will be equal to the segment of the axis between the point of contact and the focus, to which the focal tangent belongs.

Let TB be an ellipse or hyperbola, of which AB is the transverse axis, and F, O the foci, and let AH touch the ellipse, or either of the opposite hyperbolas, in the vertex A , and meet in the point H the focal tangent HG , belonging to the focus F ; the segment AH is equal to the segment AF . Fig. 6o.
61.

For let HG touch the section in T , and TF , being

BOOK drawn, will be an ordinate to AB , by the twelfth

II.

Definition. Let BG touch the section in the vertex B , and meet HG in G ; and let c be the center, and DE the conjugate axis. Then, by Cor. 2. Prop. IV. AH , DE , TF , BG are parallel; and therefore, by Cor. to Prop. XIII. Book I. $AH : BG :: HT : TG$. But it is evident, (from IO. vi.) that $HT : TG :: AF : FB$; and therefore (II. v.) $AH : BG :: AF : FB$. Consequently $AH \times BG$ is similar to $AF \times FB$, and, by Prop. IX. and Def. XI. each of these rectangles is equal to the square of CD . They are therefore equal to one another; and, as they are also similar, AH is equal to AF , and BG is equal to BF .

Cor. If OI be drawn an ordinate to AB , and on the side of AB opposite to that on which FT is, and if through I , the point in which it meets the curve, there be drawn the focal tangent KL , meeting the tangent HA in K , and the tangent GB in L ; then HL will be a parallelogram, and each of the opposite sides HK , GL will be equal to the transverse axis AB . For, by the above, and Cor. I. to Def. XI. AK , AO , BF , BG are equal to one another, and also AH , AF , BO , BL to one another. Consequently HK , GL are equal and parallel, and therefore (33. i.) HG , KL are equal and parallel. Hence HL is a parallelogram; and as AH is equal to AF , and AK to BF , HK or GL is equal to AB .

PROP. XIII.

If from any point in the curve of an ellipse or hyperbola two straight lines be drawn to the foci, their sum in the ellipse, but their difference in the hyperbola, will be equal to the transverse axis.

Fig. 60.
61.

Let P be any point in the curve of the ellipse, or hyperbola PTB , of which AB is the transverse axis, and

and the points F , o the foci, and let PO , PF , be straight lines drawn to the foci; the sum of PO , PF in the ellipse, but their difference in the hyperbola, is equal to AB .

For, the rest remaining as in the preceding Proposition and its Corollary, let PR be drawn an ordinate to AB , and let it meet the curve again in M , the focal tangent GH in N , and the focal tangent KL in Q . Then, by Cor. 2. Prop. IV. HK , OI , DE , NQ , TF , GL are parallel, and therefore, by Prop. XIII. Book I. $TH^2 : TN^2 :: AH^2 : MN \times NP$. But, on account of the parallels (10. vi.) $TH^2 : TN^2 :: FA^2 : FR^2$; and therefore (11. v.) $FA^2 : FR^2 :: AH^2 : MN \times NP$, and as, by Prop. XII. FA is equal to AH , FA^2 is equal to AH^2 , and therefore FR^2 is equal to $MN \times NP$. To these equals add the square of PR , and then (6. ii. and 47. i.) the square of NR is equal to the square of PF , and consequently NR is equal to PF . Again, by Prop. XIII. Book I. $IL^2 : IQ^2 :: BL^2 : PQ \times QM$; and on account of the parallels (10. vi.) $IL^2 : IQ^2 :: OB^2 : OR^2$, and therefore (11. v.) $OB^2 : OR^2 :: BL^2 : PQ \times QM$. But, by Prop. XII. OB is equal to BL , and therefore OB^2 is equal to BL^2 , and (14. v.) OR^2 is equal to $PQ \times QM$. To these equals add the square of RM , or its equal the square of RP , and then (6. ii. and 47. i.) the square of OM in the ellipse, and the square of OP in the hyperbola, is equal to the square of RQ . Consequently in the hyperbola OP is equal to RQ , and in the ellipse OM is equal to RQ . But in the ellipse PR , RM are equal, and the angles ORM , ORP are equal, being right angles, and OR is common to the two triangles ORM , ORP , and therefore (4. i.) OM is equal to OP . In each section therefore OP is equal to RQ , and PF to NR . Consequently in the ellipse the sum of PO , PF , but in the hyperbola their dif-

BOOK difference is equal to NA . But AN is a parallelogram,
II. and therefore (34. i.) NA , NK are equal; and as, by
 the Cor. to Prop. XII. NK is equal to AB , the sum of
 PO , PF in the ellipse, but their difference in the hy-
 perbola is equal to AB , the transverse axis.

Cor. 1. If from any point in the curve of an ellipse,
 or hyperbola, two straight lines be drawn to the foci,
 in the ellipse the difference between the transverse axis
 and either of the two will be equal to the other; but
 in the hyperbola the sum of the transverse axis and the
 least of the two will be equal to the other.

Cor. 2. If the conjugate axis DE be produced till it
 meet the opposite focal tangents in v and w , each of
 the segments cv , cw between c the center and a fo-
 cal tangent will be equal to the semitransverse axis.
 For, let the opposite focal tangents meet the transverse
 axis AB in x and y . Then, as, by Cor. 1. Def. XI. cr ,
 co are equal, it is evident, from Prop. VII. that cr ,
 cx are equal; and as xw , vy are parallel, the angles
 (29. i.) cxw , crv are equal. Consequently, as the
 angles at c are right angles, (26. i.) cv is equal to
 cw . In each section therefore the Cor. is evident.

PROP. XIV.

*If from a point without an ellipse or opposite hyperbolas
 two straight lines be drawn to the foci, their sum in the
 ellipse will be greater, but their difference in the hyper-
 bola will be less, than the transverse axis. But if from
 a point within an ellipse or hyperbola two straight lines
 be drawn to the foci, their sum in the ellipse will be less,
 but their difference in the hyperbola greater, than the
 transverse axis.*

Fig. 57.
 58.

Part I. Let E be a point without the ellipse ADB , or
 opposite hyperbolas A, B , and let EF , EO be straight
 lines

lines drawn to the foci F, O ; the sum of EF, EO in the ellipse is greater, but their difference in the hyper-
 BOOK II.

In the ellipse let EF cut the curve in D , and draw OD ; and then (20. i.) OE, ED together being greater than OD , the three OE, ED, DF together are greater than OD, DF together. Consequently, by Prop. XIII. the sum of EF, EO is greater than AB . In the hyperbolas let EF be greater than EO , and let EO cut the curve of the hyperbola in D , and draw FD . Then DF, DE together are greater than EF . But, by the Cor. to Prop. XIII. DF is equal to AB, OD together, and therefore AB, OD, DE together, or AB and OE together, are greater than EF . Consequently the difference between EF, EO is less than AB . Fig. 57. Fig. 58.

Part II. Let G be a point within the ellipse or hyperbola AD , and let the straight lines GF, GO be drawn to the foci F, O ; the sum of GF, GO in the ellipse is less, but in the hyperbola their difference is greater, than AB the transverse axis.

In either section let GF meet the curve in D , and draw DO . Then, in the ellipse, OD, DG together (20. i.) are greater than OG ; and therefore OD, DG, GF together, or OD, DF together, are greater than OG, GF together. Consequently the sum of GF, GO is less than AB , by Prop. XIII. In the hyperbola OD, DG together (20. i.) are greater than GO , and therefore OD, DG, AB together are greater than GO, AB together. But, by Cor. 1. to Prop. XIII. OD, AB together are equal to DF , and therefore DF, DG together, or FG , are greater than GO, AB together. Consequently the difference between FG, GO is greater than AB .

Cor. 1. From this and Prop. XIII. it is evident, that two straight lines being drawn from a point to the foci
 of

BOOK
II.

of an ellipse or hyperbola, if in the ellipse their sum be greater, or in the hyperbola their difference be less, than the transverse axis, the point will be without the section. If in the ellipse the sum of the two lines, or in the hyperbola their difference, be equal to the transverse axis, the point will be in the curve of the section. Lastly, if in the ellipse the sum of the two lines be less, or in the hyperbola their difference be greater, than the transverse axis, the point will be within the section.

Fig. 59. Cor. 2. If O, F be the foci of the hyperbola BP , and if the side OD , of the triangle ODF , be equal to AB , the transverse axis, and ODF be an obtuse angle, then the straight line OD produced will meet the curve of the hyperbola BP , in which the focus F is situated. For let OD be produced to K , and make the angle DFL equal to the angle FDK . Then, as by hypothesis ODF is an obtuse angle, KDF is an acute angle, and therefore, as the angle DFL is equal to it, the straight lines DK, FL , being produced, will meet. Let them meet in P , and (6. i.) PF will be equal to PD . Consequently, as the difference of PO, PF is equal to OD , or AB , the point P is in the curve of the hyperbola, by the preceding Corollary.

PROP. XV.

If from any point in the curve of an ellipse, or hyperbola, two straight lines be drawn to the foci, the straight line bisecting the angle adjacent to that contained by them will touch the ellipse; but the straight line bisecting the angle contained by them will touch the hyperbola.

Fig. 62.
63.

From the point P , in the curve of the ellipse or hyperbola BP , let two straight lines PF, PO , be drawn to the foci F, O , and in the ellipse let OP be produced to D ; the straight line PE bisecting the angle FPO ,
ad-

e 76.

asire sc

50.

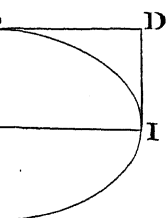


Fig. 53.

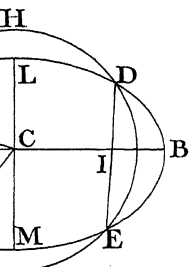


Fig. 54.

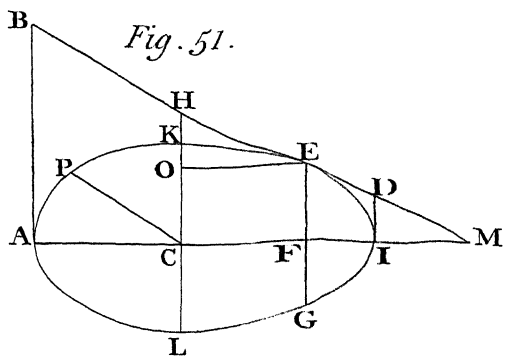
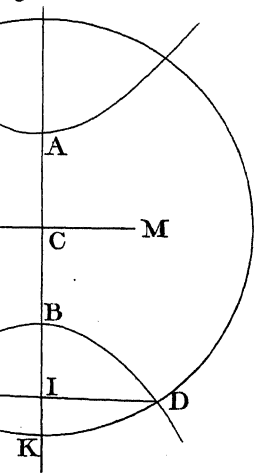


Fig. 51.

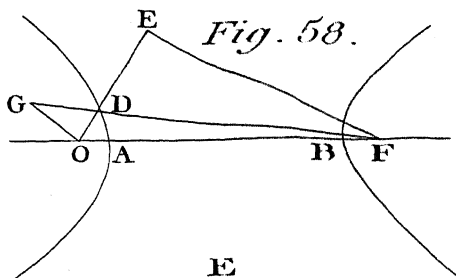


Fig. 58.

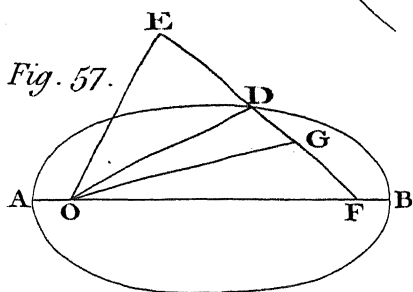


Fig. 57.

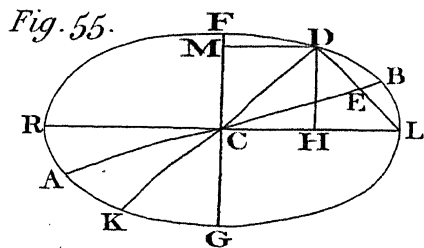


Fig. 55.

Fig. 52.

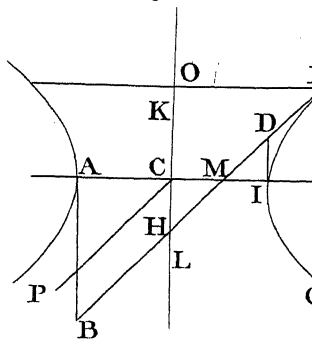


Fig. 56.

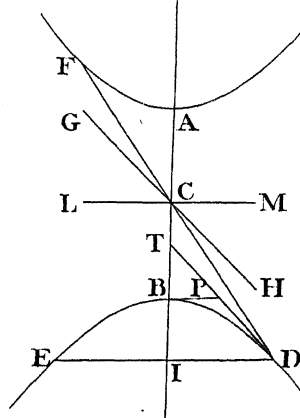
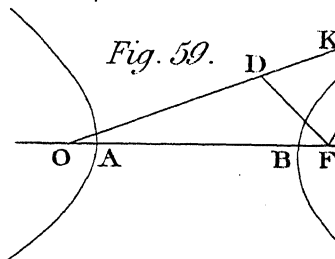
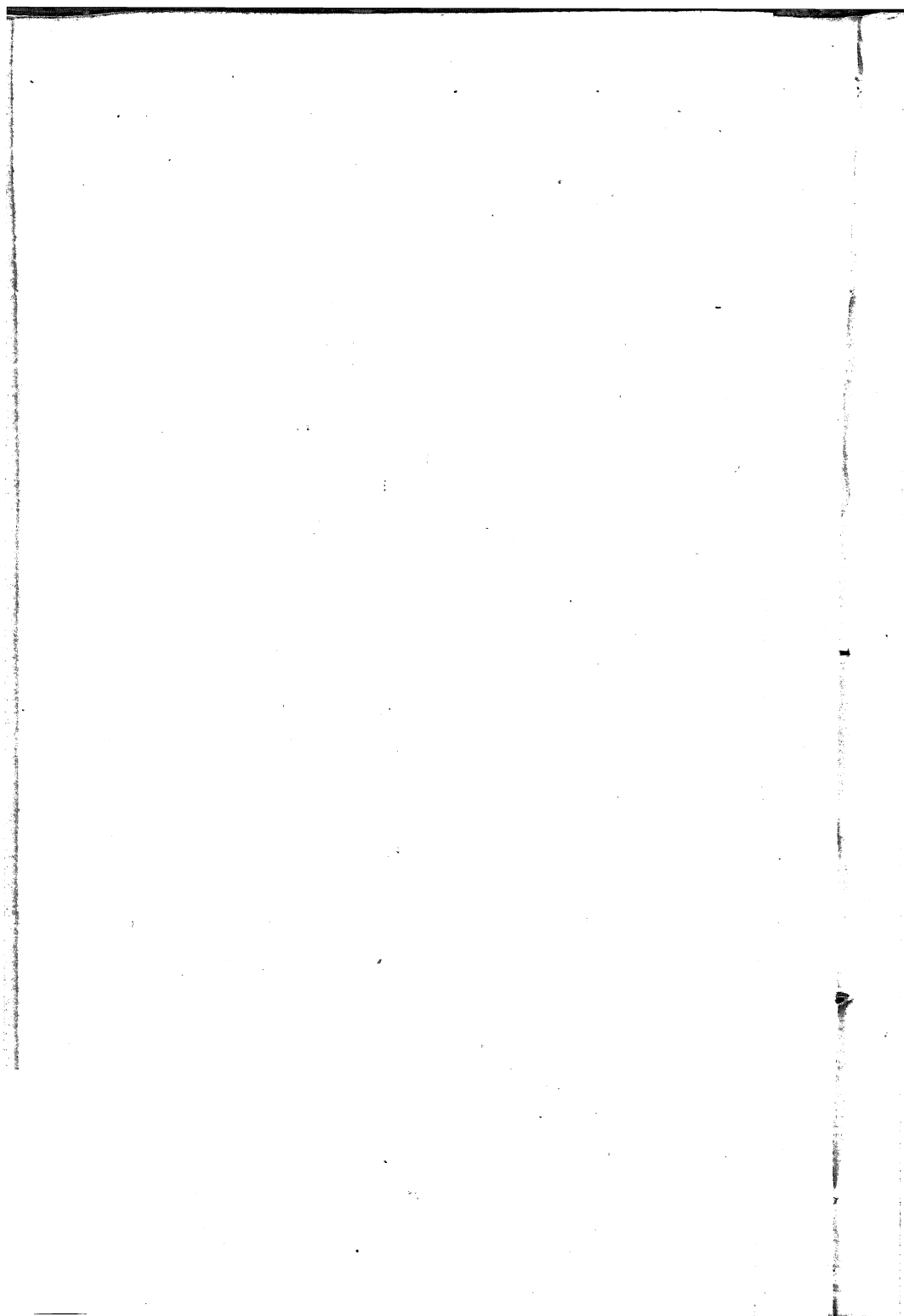


Fig. 59.





adjacent to FPO in the ellipse, touches the ellipse; but the straight line PE , bisecting the angle FPO in the hyperbola, will touch the hyperbola. BOOK II.

In the ellipse let PD , the part of OP produced, be equal to PF . Draw FD , and let it meet PE in E . Then, as FP , PD are equal, and PE common to the two triangles PEF , PED , and the angle FPE equal to the angle DPE , the side FE (4. i.) is equal to DE , and the angles FEP , DEP are equal. In EP take any point G , and draw GO , GF , GD . Then, as FE is equal to ED , and as the angles FEG , DEG are equal, we have FG (4. i.) equal to DG . But (20. i.) DG , GO together are greater than DO , or OP , PF together; and therefore, by Prop. XIII. DG , GO together are greater than AB the transverse axis. Consequently GO , GF together are greater than AB , and therefore, by Cor. 1. to Prop. XIV. the point G is without the ellipse BF , and consequently PE touches it in E . Fig. 62.

In the hyperbola take PD in PO equal to PF . Draw FD , and let it meet PE in E . Then, as PF , PD are equal, and as PE is common to the two triangles PEF , DPE , and as the angles FPE , DPE are equal, the side FE (4. i.) is equal to the side ED , and the angles FEP , DEP are also equal. In EP take any point G , and draw GO , GD , GF . Then, as FE , ED are equal, and as the angles FEG , DEG are also equal, and EG common to the two triangles FEG , DEG , the side FG (4. i.) is equal to the side DO . Also, as PD is equal to PF , by Prop. XIII. DO is equal to AB , the transverse axis. But GD , DO together (20. i.) are greater than GO , and therefore GF and AB together are greater than GO . Consequently the difference between GO and GF is less than AB , and therefore, by Cor. 1. to Prop. XIV. the point G is without the hyperbola, and PE touches the hyperbola. Fig. 63.

Cor.

BOOK
II.

Cor. 1. From this Prop. and Prop. VI. Book I. it is evident, that if a straight line touch an ellipse or hyperbola, and straight lines be drawn from the point of contact to the foci, in the ellipse the tangent will bisect the angle adjacent to that contained by these two straight lines drawn to the foci; but in the hyperbola the tangent will bisect the angle contained by these

Fig. 62. two straight lines drawn to the foci. In the ellipse the angle $\angle O P G$ (15. i.) is equal to the angle $\angle F P E$.

Fig. 62. *Cor. 2.* If from the foci O, F of an ellipse, or hyper-
63. bola, two straight lines, OD, FD be drawn to a third point D , of which OD , one of them, is equal to the transverse axis AB , and if the other FD be bisected in E , by a straight line PE at right angles; the perpendicular PE will somewhere touch the section, provided, in the hyperbola, $\angle O D F$ be an obtuse angle. And, on the contrary, if PE touch the section and bisect FD in E at right angles, then OD will be equal to the transverse axis. This is evident from Cor. 2. Prop. XIV. Prop. VI. Book I. and the above demonstration.

Fig. 64. *Cor. 3.* The rest remaining as above, let FG be at
65. right angles to the straight line LG , touching the ellipse or hyperbola in L , and let FG be produced to H , so that GH may be equal to FG ; then a straight line, bisecting DH at right angles, will pass through the focus O . For, by the preceding Cor. OH is equal to the transverse axis, and consequently equal to OD . If therefore OK be drawn, bisecting DH in K , the angles (8. i.) $\angle OKD, \angle OKH$ will be equal. Hence the Cor. is evident.

Fig. 66. *Cor. 4.* The rest remaining as in the demonstration of
67. the Proposition, let DF be so divided in L , that DL may be to LF as AB to OF , or, which is the same thing, as DO to OF , and in DF , produced as in the figures, let DN be to NF as AB to OF , and then a circle described upon

L N

re 78.

/

/

47e 57

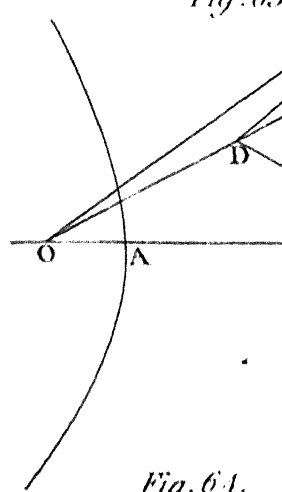
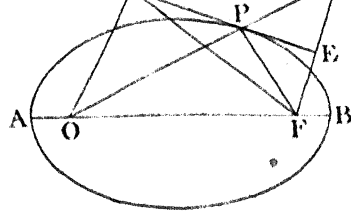
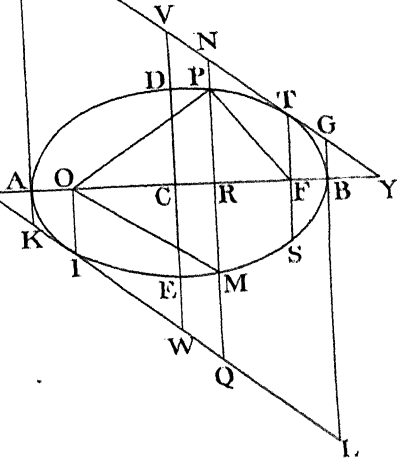


Fig. 64.

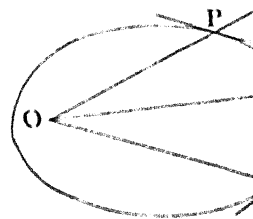


Fig. 65.

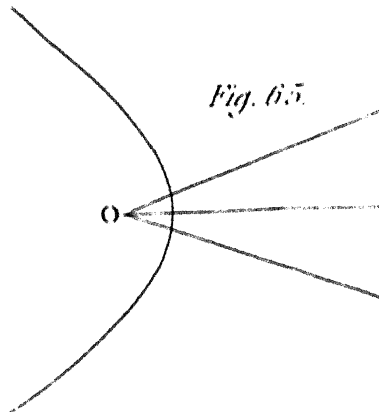
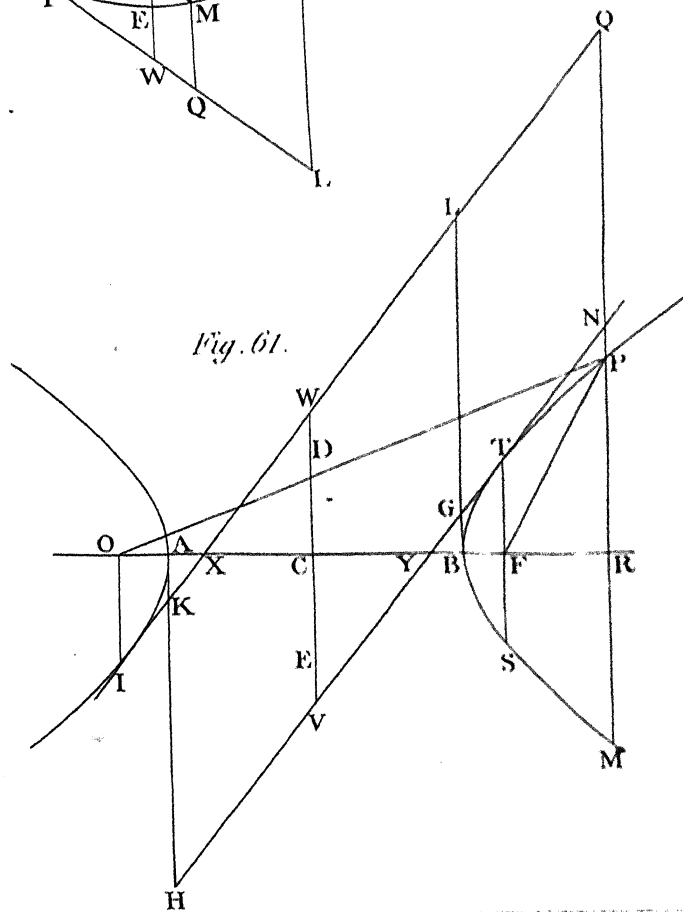
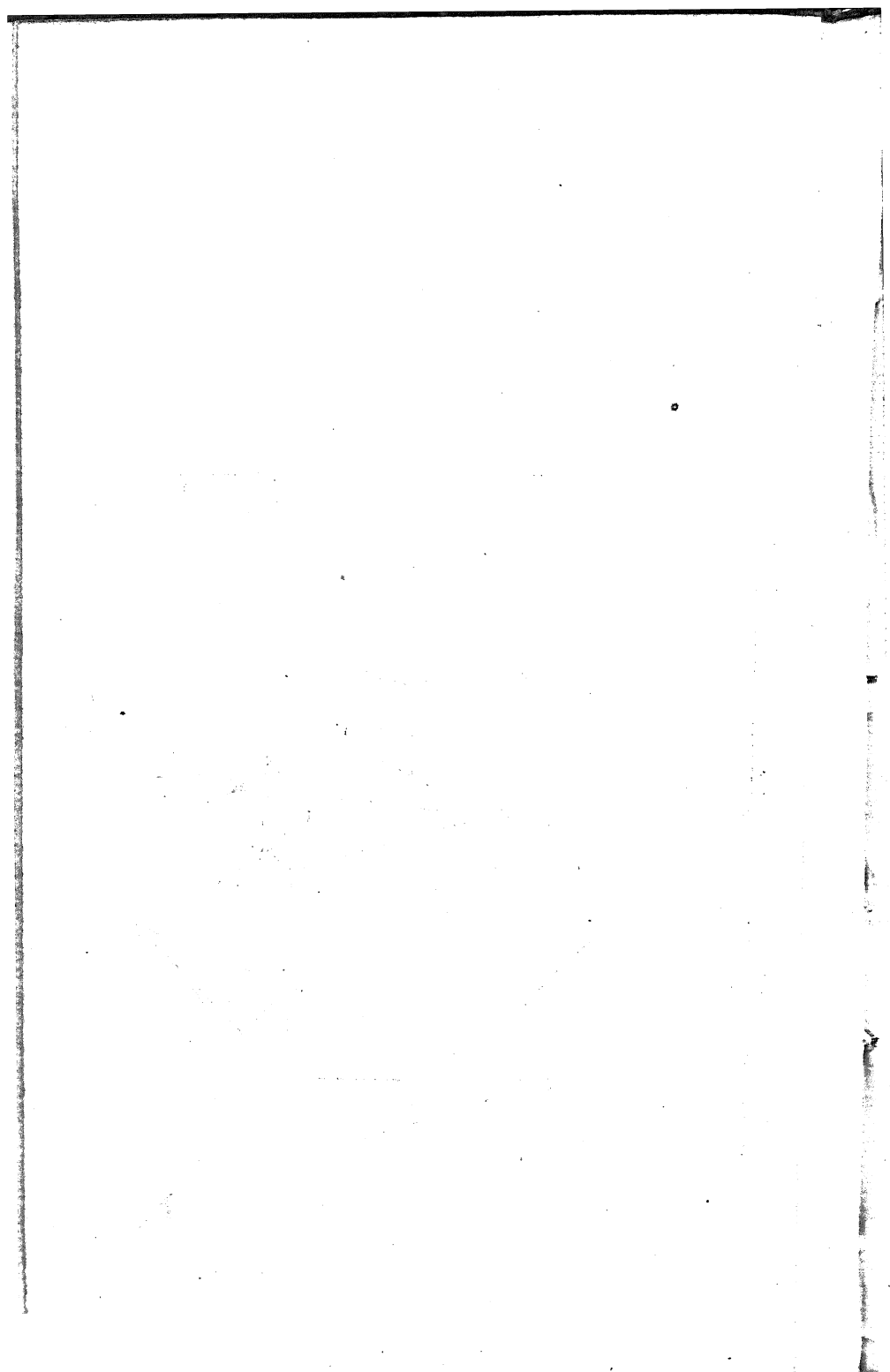


Fig. 61.





LN as a diameter will pass through the focus o . For in the ellipse produce DO to M , but in the hyperbola produce FO to M , and draw OL , ON . Then the angles DOF , FOM * (3. vi.) in the ellipse are bisected by the straight lines OL , ON ; and as the angles DOF , FOM together (13. i.) are equal to two right angles, the angles LOF , FON together are equal to a right angle. But in the hyperbola the angles DOF , DOM * are bisected (3. vi.) by the straight lines OL , ON ; and as the angles DOF , DOM together are equal to two right angles, the angles LOD , DON together are equal to a right angle. In either case therefore the angle LON is a right angle, and consequently (31. iii. and 21. i.) a circle described about LN as a diameter will pass through o .

Sir Isaac Newton makes much use of the properties expressed in the three last Corollaries. See the Principia, Sect. IV. Book I.

PROP. XVI.

If a straight line touching an ellipse or hyperbola meet a straight line drawn from either of the foci, and be at right angles to it, the straight line joining the center and the point of concurrence will be equal to the semitransverse axis: or, if a straight line touch an ellipse or hyperbola, and straight lines be drawn from the point of contact to the foci, a straight line drawn from the center to the tangent, and parallel to either of the two drawn to the foci, will be equal to the semitransverse axis.

Part I. Let the straight line GP , touching the ellipse or hyperbola PB in the point P , meet in the points G ,

Fig. 68.
69.

* That the angle FOM in the ellipse, and the angle DOM in the hyperbola, is bisected by ON , is proved by Simson, in his Prop. A. in the sixth Book of his edition of Euclid.

draw CG , CE to the points of concurrence; each of the straight lines CG , CE is equal to AC or CB , the semitransverse axis.

For, draw FP , OP , and let FP , or FP produced, meet OE produced in D . Then, as PEO is a right angle, PED is a right angle; and as PE is common to the two triangles PEO , PED , and as, by Cor. I. to Prop. XV. the angles OPE , DPE are equal, the side PD (26. i.) is equal to the side PO , and DE is equal to EO . Consequently, by Prop. XIII. FD is equal to the transverse axis AB ; and as FC , CO are equal, and DE equal to EO , $FO : CO :: DO : EO$. The straight lines FD , (2. vi.) CE are therefore parallel, and $OF : FD :: OC : CE$. But OC is the half of OF , and therefore CE is equal to the half of FD . Consequently CE is equal to AC or CB , the semitransverse axis. If FG , PO be produced till they meet in H , it may be proved, in the same manner, that FG is equal to GH , FP to PH , HO to AB , and CG to AC or CB .

Fig. 68.
69.

Part II. Let the straight line GP touch the ellipse or hyperbola PB in the point P , and let FP , PO be straight lines drawn from the point of contact to the foci F , O . Let c be the center, and draw CE parallel to FP , and CG parallel to OP , and let CE meet the tangent in E , and CG meet it in G ; each of the straight lines CE , CG is equal to AC or CB , the semitransverse axis.

For draw OE , and, being produced, let it meet FP , or FP produced, in D . Then, as FD , CE are parallel, (2. vi.) $FC : CO :: DE : EO$, and as, by Cor. to Def. XI. FC , CO are equal, DE is equal to EO . By Cor. I. to Prop. XV. the angles OPE , DPE are equal, and therefore (3. vi.) $OE : ED :: OP : PD$, and as
 OE ,

O E, E D are equal, O P is equal to P D. Consequently, BOOK
II.
by Prop. XIII. F D is equal to A B, the transverse axis ;
and as F D, C E are parallel, $FO : FD :: CO : CE$.
But c o is the half of F O, and therefore c e is the half
of F D. Consequently c e is equal to A c or C B, the
semitransverse axis. If F G be drawn, and, being pro-
duced, meet O P produced in H, it may be proved, in
the same manner, that F G is equal to G H, F P to P H,
and C G to A C or C B.

Cor. The rest remaining as above, if the straight line
I K, drawn through c the center, and parallel to the
tangent G P, meet P F in I and P O in K, the segments
P I, P K are equal ; and each of them is equal to A c
or C B. For (34. i.) P K is equal to C G and P I is equal
to C E.

The demonstrations of the 11th and 12th Proposi-
tions of the first Book of the Principia depend, in a
very considerable degree, on this property.

PROP. XVII.

*The rectangle contained under two straight lines, drawn
from the foci of an ellipse or hyperbola to a tangent, and
at right angles to it, is equal to the square of the semi-
conjugate axis. And the rectangle contained under two
straight lines, drawn from the transverse axis of an el-
lipse or hyperbola to a tangent, and at right angles to
it, is equal to the square of the semiconjugate axis, if one
of them be drawn from the center, and the other meet
the tangent in the point of contact.*

Part I. Let the straight lines F G, P E, drawn from Fig. 70.
71.
the foci F, O of the ellipse or hyperbola P B, meet in
the points G, E, the straight line G E which touches
the section in P, and let them be at right angles to the
tangent G E, and c being the center, let c d be the se-
miconjugate

BOOK II. miconjugate axis; the rectangle under FG , OE is equal to the square of CD .

Draw EC , and, being produced, let it meet FG , or FG produced, in H . Then as FG , OE are at right angles to GP , they are parallel to one another, (28. i.) and (29. i.) the angles CFH , COE are equal. The triangles (15. and 32. i.) CFH , COE are therefore equiangular, and, Cor. 1. to Def. XI. CF is equal to CO . Consequently (26. i.) CH is equal to CE , and FH is equal to OE . If therefore, with C as a center, and CA or CB as a distance, a circle be described, it will pass through the points E , G , by Prop. XVI. and consequently through H ; and therefore (35. and 36. iii.) the rectangle under GF , FH is equal to the rectangle under AF , FB . But as FH is equal to OE , the rectangle under GF , EO is equal to the rectangle under AF , FB ; and as, by the eleventh Definition, the rectangle under AF , FB is equal to the square of CD , the rectangle under GF , EO is also equal to the square of CD .

Part II. Let AB be the transverse axis of the ellipse or hyperbola PB , of which C is the center, and let GE touch the section in the point P ; the rectangle under the straight lines CK , MP , drawn from the transverse axis AB to the tangent GE , and at right angles to it, is equal to the square of CD , the semiconjugate axis.

Let the conjugate axis meet the tangent in the point I ; and from the point P draw PN an ordinate to AB , and PL an ordinate to DCI . Then, as CK , MP are at right angles to the tangent GE , they are (28. i.) parallel to one another, and, by Cor. 2. Prop. IV. as PN is an ordinate to AB it is parallel to DCI , and as PL is an ordinate to DCI it is parallel to AB . Consequently (29. i.) the angle KCB is equal to the angle PMN ; and as ICB , PNM are right angles, the angles ICK ,

KCB

$\angle C B$ together are equal to the angles $\angle M P N$, $\angle P M N$ together, and therefore the angles $\angle I C K$, $\angle M P N$ are equal, and the triangles $\triangle I C K$, $\triangle M P N$ are equiangular. Hence $CK : CI :: PN : PM$. But (34. i.) PN is equal to CL , and therefore $CK : CI :: CL : PM$, and (16. vi.) $CK \times PM$ is equal to $CI \times CL$. Consequently, as the rectangle under CI , CL , by Prop. VII. (and 17. vi.) is equal to the square of CD , the rectangle under CK , PM is also equal to the square of CD .

PROP. XVIII.

If a straight line touching an ellipse or hyperbola be limited by tangents passing through the vertices of the transverse axis, the circumference of a circle described about it as a diameter will pass through the foci; and the rectangle under the two straight lines drawn from the point in which it touches the section to the foci will be equal to the square of the semidiameter parallel to it.

Part I. Let the straight line GE touch the ellipse or hyperbola FB in the point F , and meet in the points G , E the tangents AG , BE , passing through the vertices A , B of the transverse axis AB ; the circumference of a circle described about GE as a diameter will pass through the foci F , O .

Fig. 72.
73.

For, by the seventh Definition, and Cor. 2. Prop. IV. $\angle GAB$, $\angle EBA$ are right angles; and, by the eleventh Definition, and Prop. IX. the rectangle under AG , BE is equal to the rectangle under AO , OB . Consequently (17. vi.) $EB : BO :: AO : AG$, and therefore the straight lines EO , GO being drawn, the angle $\angle EOB$ (6. vi.) is equal to the angle $\angle AGO$, and the angle $\angle BEO$ is equal to the angle $\angle AOG$. The angles $\angle EOB$, $\angle AOG$ together are therefore equal to a right angle, and consequently (32. i.) the angle $\angle GOE$ is a right

BOOK
II.

right angle. If therefore a circle be described about GE as a diameter, it is evident (from Prop. 31. iii. and 21. i.) that its circumference must pass through the focus O ; and in the same way it may be proved, that it must pass through the focus F .

Part II. The rest remaining as above, let PO , PF be drawn to the foci O , F , and let CD be the semidiameter parallel to GE ; the rectangle under PF , PO is equal to the square of CD .

For, draw OI perpendicular to GE , and, being produced, let it meet PF , or PF produced, in H . Then, by Cor. 1. to Prop. XV. the angles OPI , HPI are equal, and the angle OIP is equal to the angle HIP , each of them being a right angle, and PI is common to the two triangles OPI , HPI . Consequently (26. i.) PO is equal to PH , and OI to IH ; and it is evident (from 3. iii.) that the point H must be in the circumference of the circle described about GE as a diameter, and passing through F , O , according to Part I. The rectangle under FP , PH , or that under FP , PO , (35. and 36. iii.) is therefore equal to the rectangle under GP , PE . Consequently, by Prop. IX. the rectangle under FP , PO is equal to the square of CD .

PROP. XIX.

A straight line drawn from either of the foci of an ellipse or hyperbola, perpendicular to a tangent, is to a straight line drawn from the same focus to the point of contact, as the semiconjugate axis to the semidiameter parallel to the tangent.

Fig. 70.
71.

Let PB be an ellipse or hyperbola, of which the foci are F , O , and let GE touch the section in P , and let FG be perpendicular to it; the perpendicular FG is to the straight line FP , joining the focus F and point of

con-

contact, as CD the semiconjugate axis to CR the semidiameter parallel to GE .

For, draw OE perpendicular to GE , and draw PO . Then, by Cor. I. to Prop. XV. the angles FPG , OPE are equal; and FGP , OEP being right angles, the triangles FPG , OPE , are equiangular. Consequently (4. vi.) $FG : FP :: OE : OP$, and by alternation $FG : OE :: FP : OP$; and therefore (22. vi.) $FG \times OE : FP \times OP :: FG^2 : FP^2$. But, by Prop. XVII. $FG \times OE$ is equal to CD^2 , and $FP \times OP$ is equal to CR^2 , by Prop. XVIII. Consequently $CD^2 : CR^2 :: FG^2 : FP^2$, and therefore (22. vi.) $CD : CR :: FG : FP$.

Cor. I. The rest remaining as above, let CE parallel to FP meet the tangent GE in E , and let CK be perpendicular to GE and meet it in K . Then (4. vi.) $FG : FP :: CK : CR$. But, by Prop. XVI. CE is equal to CB , the semitransverse axis, and therefore, by the above, (and II. v.) $CK : CB :: CD : CR$.

Cor. 2. The rest remaining as above, let the straight line PM , perpendicular to the tangent GE , meet the transverse axis AB in M , and then CB will be to CD as CR to PM . For, by the preceding Cor. $CK : CB :: CD : CR$, and therefore (1. vi.) $CK \times PM : CB \times PM :: CD^2 : CR \times CD$. But, by Prop. XVII. $CK \times PM$ is equal to CD^2 , and therefore (14. v.) $CB \times PM$ is equal to $CR \times CD$. Consequently (16. vi.) $CB : CD :: CR : PM$.

PROP. XX.

If a straight line touch an ellipse or hyperbola, and from the point of contact two straight lines be drawn to an axis, the one an ordinate to it, the other perpendicular to the tangent, the segment of the axis between the center and ordinate will be to the segment between the perpendicular and ordinate as the axis to its parameter.

BOOK
II.

Fig. 74.
75.

Let the straight line IR touch the ellipse PB in the point P ; and AB being the major axis, and DE the conjugate axis, and c the centre, let ce be an ordinate to AB , and PG an ordinate to DE . Let the straight line MP be perpendicular to AB in K and DE in M ; then CH is to AB as its parameter, and CG is to GM as its parameter.

Let the tangent IR meet AB in R , and DE in I . Then, by Cor. 2. Prop. VII. the rectangle HR is equal to the rectangle under AI ; RPK being a right angle, and PH being perpendicular to KR , the rectangle under KH , HR is equal to the square of PH (Cor. 8. vi. and 16. vi.).
 $CH : KH :: CH \times HR : KH \times HR$; and on account of the equals, (and II. v.) $CH \times HR : KH \times HB : PH^2$. Consequently, as $CH \times HR$ is to PH^2 as AB to its parameter CH (II. v.) as AB to its parameter.

Again (I. vi.) in the ellipse, $CG : GM : GM \times GI$; and as above, by Cor. 2. Prop. VII. the rectangle under CG , GI is equal to the rectangle under EG , GD , and (Cor. 8. vi.) the rectangle under GM , GI is equal to the square of PG .
 $CG : GM :: EG \times GD : PG^2$; and the Prop. VI. (and II. v.) CG is to GM as D to its parameter.

Lastly, in the hyperbola, as, by Cor. 2. Prop. VII. PG is parallel to AB , and PH parallel to DE , GM as KP to PM , and therefore as KH to PM (I. vi.) $KH : HC :: KH \times RH : HC \times PM$ (Cor. 8. vi. and 17. vi.) $KH \times RH$ is equal to the square of PH , and, by Cor. 2. Prop. VII. $HC \times RH$ is equal to the square of PH . Consequently (II. v.) $CG : GM : PH^2$ is to PH^2 as AB to its parameter. But, by Def. VIII. PH^2 is to

66.

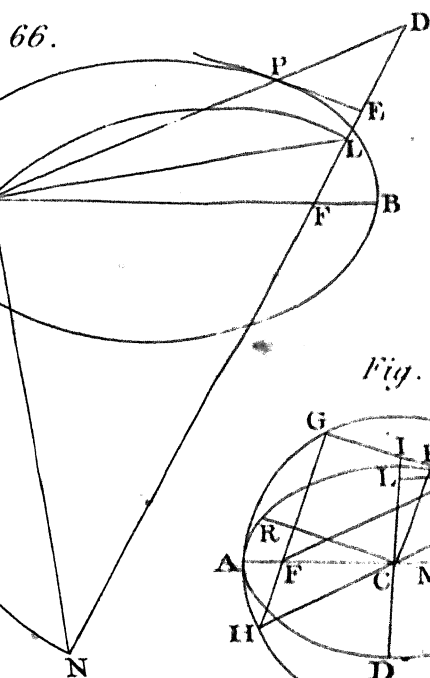


Fig. 70.

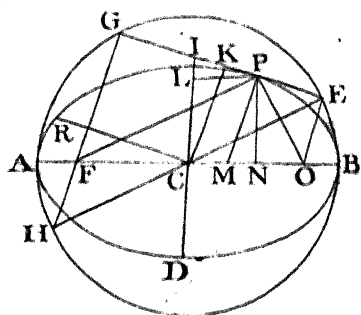


Fig. 67.

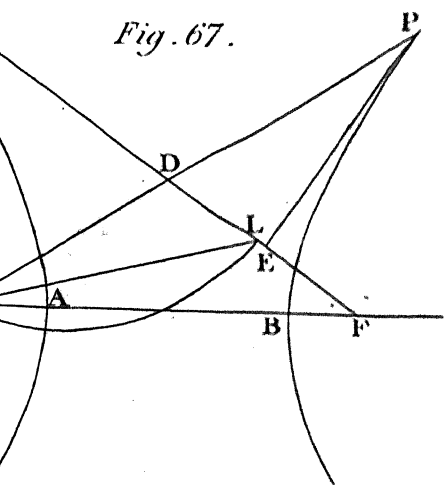


Fig. 68.

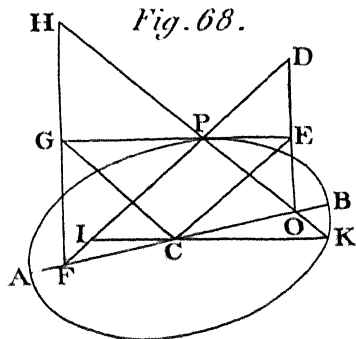


Fig. 69.

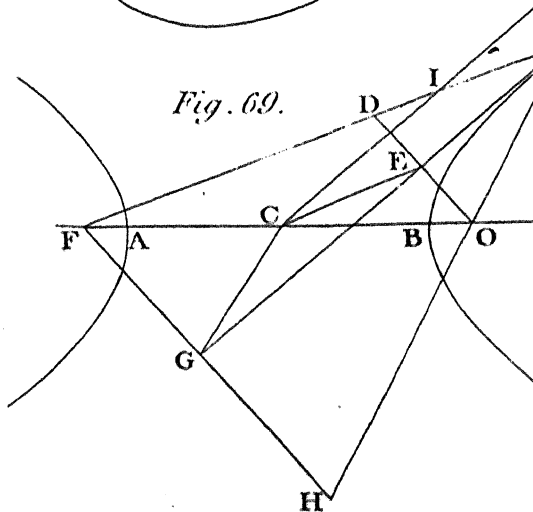
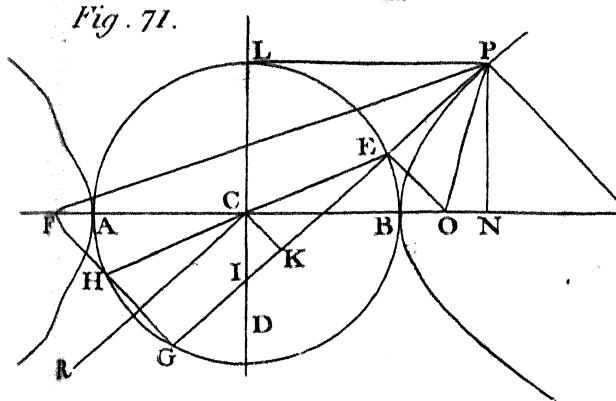
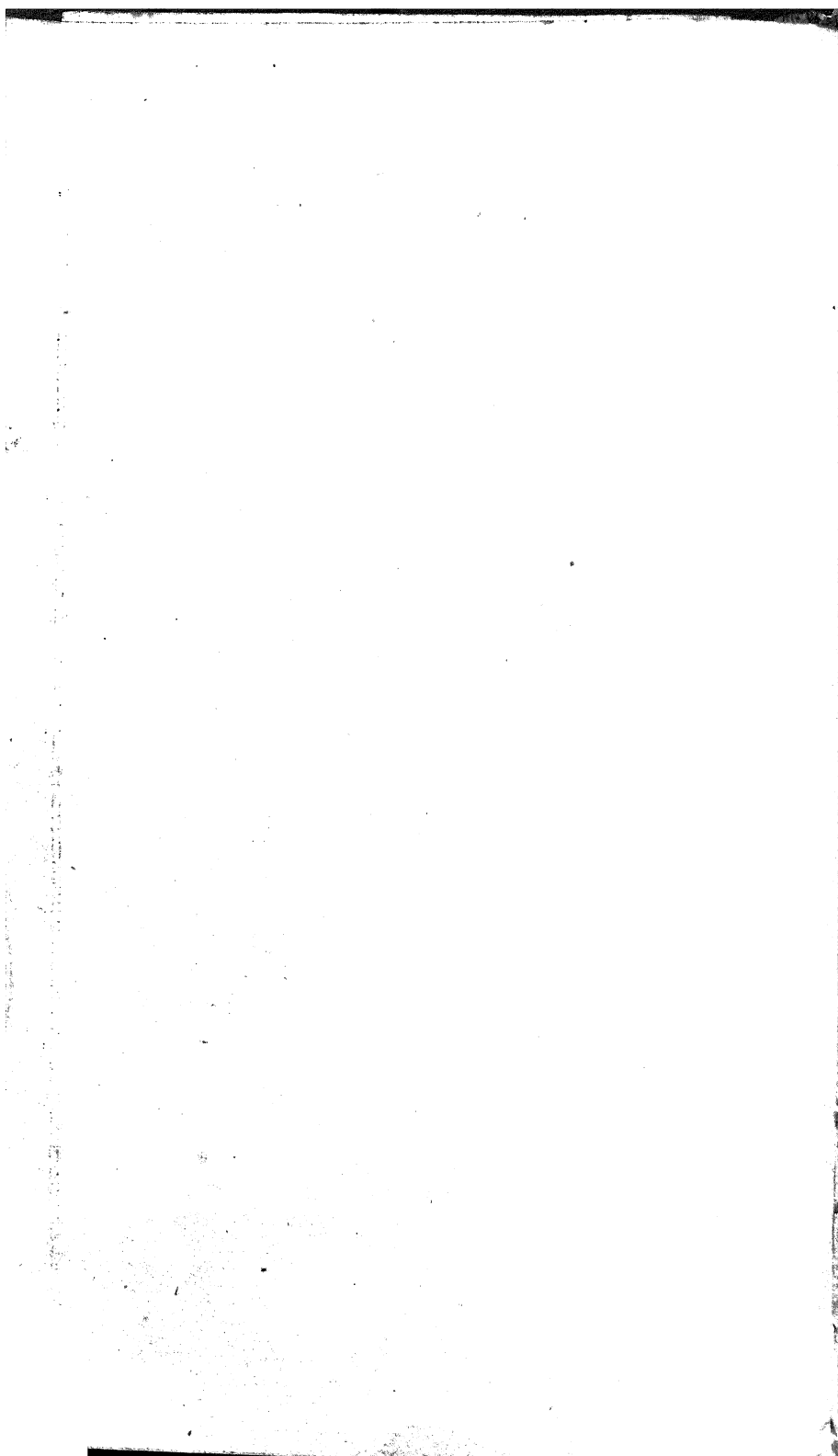


Fig. 71.





is evident from Def. IX. DE is to AB as DE is to its parameter. Consequently CG is to GM as DE is to its parameter.

PROP. XXI.

If a straight line touch an ellipse or hyperbola, and a straight line be drawn from the point of contact at right angles to it, and meet the axes, the rectangle under the segments of the perpendicular, between the point of contact and the axes, will be equal to the square of the semidiameter parallel to the tangent.

Let the straight line IR touch the ellipse or hyperbola PB in the point P , and let the straight line MP , at right angles to IR , meet the transverse axis AB in K , and the conjugate axis DE in M , and, C being the center, let CL be the semidiameter parallel to IR ; the rectangle under PM , PK is equal to the square of CL .

For, by Cor. 2. Prop. XIX. and inversion, $PK : CL :: CD : CB$, and therefore $PK^2 : CL^2 :: CD^2 : CB^2$. And PG being drawn an ordinate to DE , by Prop. XX. CG is to GM as DE to its parameter. But, by Def. IX. (and Cor. 2. to 20. vi.) DE is to its parameter as DE^2 to AB^2 , or (15. v.) as CD^2 to CB^2 ; and as PG is parallel to AB , (2. vi.) $CG : GM :: PK : PM$. Consequently (11. v.) $PK : PM :: CD^2 : CB^2$; and therefore (1. vi.) $PK^2 : PM \times PK :: CD^2 : CB^2$. But, by the above, $PK^2 : CL^2 :: CD^2 : CB^2$; and therefore (11. v.) $PK^2 : PM \times PK :: PK^2 : CL^2$. Consequently (14. v.) $PM \times PK$ is equal to CL^2 .

Cor. By the above, and Prop. XVIII. the rectangle under PM , PK is equal to the rectangle under PO , PF , the straight lines drawn from P the point of contact to the foci O , F .

Fig. 74.
75.



BOOK
II.

PROP. XXII.

If a circle be described about the transverse axis of an ellipse as a diameter, a polygon may be inscribed in it, and a corresponding polygon in the ellipse, so that the polygon in the circle shall be to that in the ellipse as the transverse axis to the conjugate axis.

FIG. 76.

Let AB be the transverse, and FG the conjugate axis of the ellipse $AFBG$, and about AB as a diameter let the circle $ADBE$ be described; a polygon may be inscribed in $ADBE$, and a corresponding one in the ellipse $AFBG$, so that the polygon in the circle shall be to that in the ellipse as AB to FG .

For, let c be the common center of the ellipse and circle, and produce FG till it meet the circumference of the circle in D, E . Let K, M be points in the circumference, and draw DK, KM, MA . Draw KP, MO parallel to DC , and let them meet AC in P, O , and the curve of the ellipse in L, N , and draw FL, LN, NA . Draw KH, LI parallel to AB , and let them meet DC in H, I . Then PI, PH are parallelograms, and (34. i.) CH is equal to PK , and CI equal to PL ; and KP, MO are perpendicular to AC , and LP, NO are ordinates to AB , by Cor. 2. Prop. IV. By Prop. V. $AC^2 : CF^2 :: AP \times PB : PL^2$, and therefore (35. iii.) as $AP \times PB$ is equal to KP^2 , $AC^2 : CF^2 :: KP^2 : PL^2$, and (22. vi.) AC or $DC : CF :: KP$ or $CH : PL$ or CI^* . Consequently (19. v.) $DC : FC :: DH : FI$; and $DH : FI :: CH : CI$. But (I. vi.) $CH : CI ::$ parallelogram $PH : parallelogram PI ; and $DH : FI ::$ the triangle $DKH : the triangle FLI . Consequently$$

* This is the property referred to by writers on the Orthographical Projection of the Sphere, when they prove, that a circle, not parallel to the plane of projection, is projected into an ellipse.

(II. and 12. v.) $DC : FC ::$ the trapezium $DKPC$: BOOK II.
 the trapezium $FLPC$. In the same manner it may be
 demonstrated that $DC : FC ::$ the trapezium $KMOP$:
 the trapezium $LNOP$; and also that $DC : FC ::$ the
 triangle MAO : the triangle NAO . But (15. v.) $DC :$
 $FC :: AB : FG$, and therefore (12. v.) $AB : FG ::$ the
 polygon $DKMAC$: the polygon $FLNAC$. Conse-
 quently, as inscriptions may be made in a similar man-
 ner all round the circle and ellipse, the Prop. is evident.

Cor. A polygon may be inscribed in an ellipse, which
 shall be deficient from the ellipse by a superficies less
 than any given superficies. For, the rest remaining as
 above, if a straight line parallel to LF be drawn to
 touch the ellipse, and meet CF and PL produced, and
 from F, L straight lines be drawn to the point of con-
 tact, the triangle thus formed will be equal to half the
 parallelogram contained by LF , the tangent parallel
 to it, and CF, PL produced. This triangle therefore
 will be greater than half the elliptic segment contained
 under the curve LF , and the straight line LF . Such
 an inscription therefore being made all round the el-
 lipse, the *Cor* is evident (from 1. x.).

PROP. XXIII.

If the transverse axis of an ellipse be also a diameter of a circle, the ellipse will be to the circle as the conjugate axis to the transverse axis.

Let AB be the transverse axis of the ellipse $AFBG$,
 and also a diameter of the circle $ADBE$; the ellipse is
 to the circle as FG the conjugate axis to AB the trans-
 verse axis.

For, every thing remaining as in Prop. XXII. let the Fig. 76.
 circle QRS be to the circle $ADBE$ as FG to AB ;
 and then, if it can be proved that the circle QRS is
 equal

BOOK
II.

equal to the ellipse $A F B G$, the truth of the Proposition will be manifest. If the circle $Q R S$ be not equal to the ellipse, let it first, if possible, be greater. Then it is possible to inscribe in the circle $Q R S$ a polygon, having an even number of sides, and greater than the ellipse $A F B G$. Let it be understood to be inscribed, and let a polygon similar to it be supposed to be inscribed in the circle $A D B E$; and from the angular points of the polygon in $A D B E$ let straight lines be drawn parallel to $D E$. Let the points in which these parallel lines cut the curve of the ellipse be joined, and then a polygon will be inscribed in the ellipse corresponding to the polygon in the circle $A D B E$, as in the last Proposition; and $A B : F G ::$ the polygon inscribed in the circle $A D B E$: the polygon inscribed in the ellipse. But, by hypothesis and inversion, the circle $A D B E$: the circle $Q R S :: A B : F G$; and therefore (II. v.) the polygon inscribed in the circle $A D B E$: the polygon inscribed in the ellipse :: the circle $A D B E$: the circle $Q R S$. Consequently (I. and 2. xli.) the polygon inscribed in the circle $A D B E$: the polygon inscribed in the ellipse :: the polygon inscribed in the circle $A D B E$: the polygon inscribed in the circle $Q R S$. The polygon (I. v.) inscribed in the ellipse is therefore equal to the polygon inscribed in the circle $Q R S$: which is absurd; for, by the present hypothesis, the polygon inscribed in the circle $Q R S$ is greater than the ellipse.

Secondly, if it be possible, let the circle $Q R S$ be less than the ellipse. Then it is possible, by Cor. Prop. XXII: to inscribe in the ellipse a polygon greater than the circle $Q R S$, and a polygon corresponding to it in the circle $A D B E$; and to inscribe in the circle $Q R S$ a polygon similar to the polygon inscribed in the circle $A D B E$. Let such polygons be supposed to be so inscribed.

lygon inscribed in the circle QRS , contrary to the construction which has now been supposed to be made. The circle QRS therefore is equal to the ellipse $AFBG$, and therefore the ellipse $AFBG$ is to the circle $ADBE$ as FG to AB .

Cor. 1. An ellipse is equal to a circle, whose diameter is a mean proportional between its axes. For, by the above, $FG : AB ::$ the ellipse $AFBG$, or the circle QRS : the circle $ADBE$. But (I. vi.) $FG : AB :: FG \times AB : AB^2$; and (2. xii.) the circle QRS : the circle $ADBE :: QS^2 : AB^2$, QS being the diameter of the circle QRS . Consequently (II. v.) the ellipse $AFBG$: circle $ADBE :: QS^2 : AB^2 :: FG \times AB : AB^2$: and (14. v.) QS^2 is equal to $FG \times AB$.

Cor. 2. From the preceding (and 17. vi. and 2. xii.) it is evident, that the areas of two ellipses are to one another as the rectangles under their axes.

The *Cor.* to Prop. XIV. Lib. I. of the Principia depends, in a great degree, upon this truth.

PROP. XXIV.

If from a point in the conjugate axis of an ellipse a straight line, equal to the difference of the semiaxes, be drawn to a point in the transverse axis, and be produced beyond the transverse axis, so that the part produced be equal to the semiconjugate axis, the extremity of the part produced will be in the curve of the section. Or, if from a point in the conjugate axis of an ellipse a straight line, equal to the sum of the semiaxes, be drawn to a point in the transverse axis, and if this line be so cut that the segment between the transverse axis and the
point



BOOK
II.

point of section be equal to the semiconjugate axis, the point of section will be in the curve.

Fig. 77.
78.

Let $A D B E$ be an ellipse, of which $A B$ is the transverse, and $D E$ is the conjugate axis, and c is the center. Let F be a point in the conjugate, and G be a point in the transverse axis, and let the straight line $F G$ be equal to the difference or sum of $c B$, $c D$. When $F G$ is equal to the difference of $c B$, $c D$, as in Fig. 78. let $F G$ be produced beyond $A B$ to H , so that $G H$ be equal to $c D$; but when $F G$ is equal to the sum of $c B$, $c D$, as in Fig. 77. let the segment $G H$ be equal to $c D$; and in either case the point H will be in the curve of the ellipse.

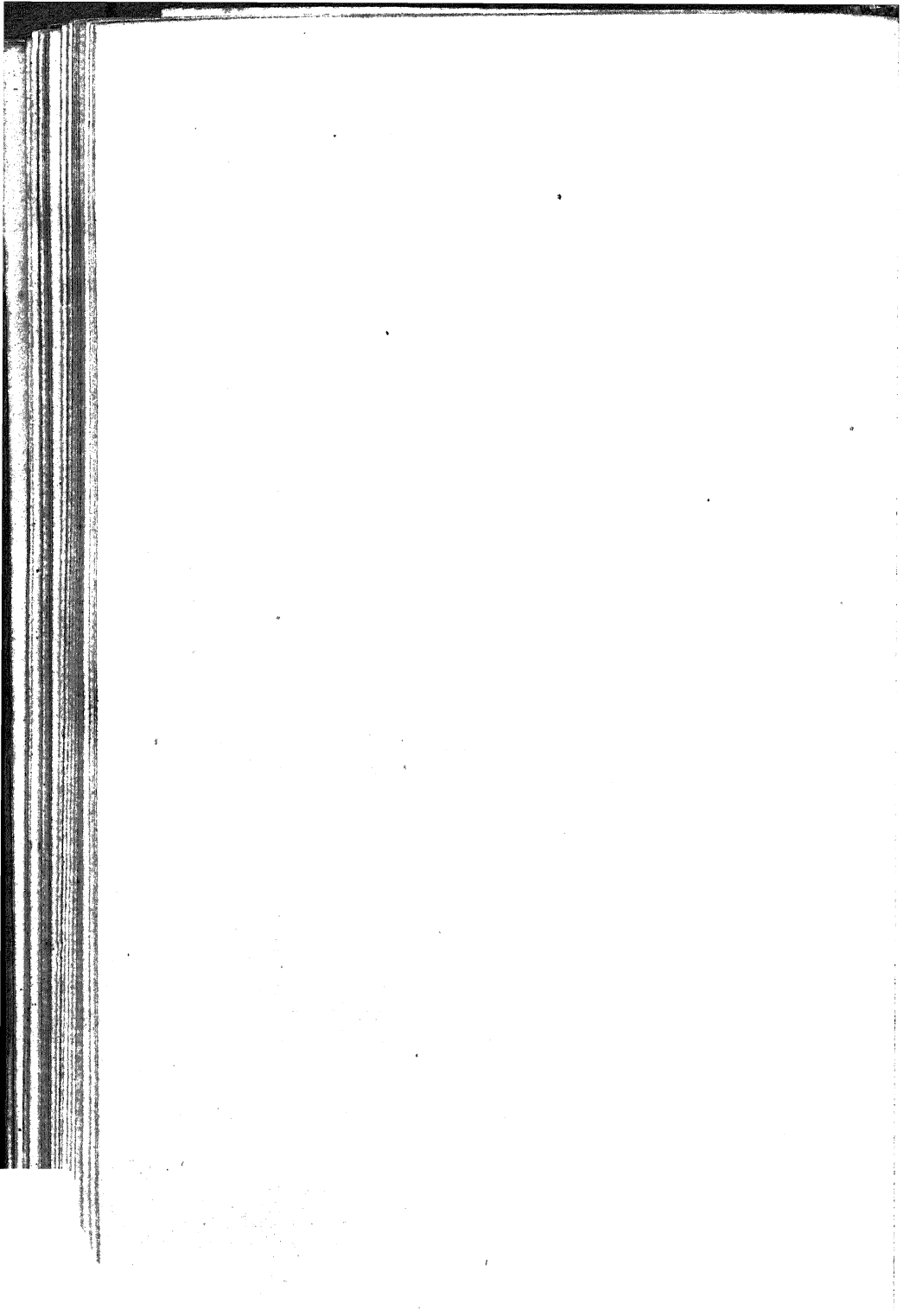
For through the center c draw the straight line $c K$ parallel to $F H$. Through H draw the straight line $H K$ parallel to $D E$, and let it meet $c K$ in K , and $A B$ in I . Then (34. i.) the straight line $c K$ is equal to $F H$. But as $F G$ is equal to the sum or difference of $c B$, $c D$, and as $G H$ is equal to $c D$, the straight line $F H$ is equal to $c B$; and consequently $c K$ is equal to $c B$. With c , therefore, as a center, and $c B$ as a distance, let a circle be described, and it will pass through K . Again, on account of the similar triangles $c K I$, $G H I$, $c K^2 : G H^2 :: K I^2 : H I^2$; and therefore, on account of the equals, $c B^2 : c D^2 :: K I^2 : H I^2$. But (3. and 35. iii.) the square of $K I$ is equal to the rectangle under $A I$, $I B$; and therefore $c B^2 : c D^2 :: A I \times I B : H I^2$. The point H is therefore in the curve, by Cor. 1. Prop. V. (and 9. v.) for $H I$ is parallel to the ordinates of $A B$.

SCHOLIUM.

The instrument called by some *the trammels*, and by others

others the *elliptic compasses*, used by cabinet-makers, &c. for describing the curves of ellipses, are constructed on the property demonstrated in this Proposition. As the trammels are in general use, it is needless to give a description of them in this place. Lathes for making picture-frames, and ornaments of an elliptical form, are constructed on the same property.

BOOK
II.



A
GEOMETRICAL TREATISE
OF
CONIC SECTIONS.

BOOK III.

Of the Parabola, the Directrices of the Sections, the Asymptotes of the Hyperbola, Conjugate Hyperbolas, and of hyperbolic Sectors and Trapezia.

DEFINITIONS.

I.

THE section FDC being a parabola, and VBE its vertical plane, as in the 15th and 16th Definitions in the first Book, any straight line, as DI , in the parabola parallel to VB , the side in which VBE touches the cone, is called a *Diameter* of the parabola. Fig. 79.

Cor. 1. From this Definition (and 9. xi.) the diameters of a parabola are parallel to one another; and, by Prop. XIV. Book I. any straight line drawn in the plane of a parabola, parallel to a diameter, will meet the curve in one point, and in one point only, and, by this Definition, it will itself be a diameter.

Cor.

BOOK
III.

Cor. 2. Any straight line in a parabola, not parallel to a diameter, will meet the curve in two points. For any straight line drawn through v , the vertex of the cone, and in the vertical plane, and not in the same direction with vB , will fall without the opposite cones; and by the demonstration of the second part of Prop. VIII. of the first Book, one plane may be drawn through this straight line to touch the conical superficies, and cut the plane of the parabola. The intersection also of this plane with the plane of the parabola will touch the parabola, and any straight line in the parabola parallel to this tangent will meet the curve in two points, by *Cor.* 1. Prop. VIII. Book I. Hence (16. xi.) the *Cor.* is evident.

II.

The point in which a diameter of a parabola meets the curve is called the *Vertex* of the diameter.

III.

If a straight line terminated by the curve of a parabola be bisected by a diameter, it is called a *Double Ordinate* to that diameter; and its half is simply called an *Ordinate* to it.

IV.

The segment of a diameter between its vertex and an ordinate is called an *Absciss* of that diameter.

V.

The diameter of a parabola, which cuts its ordinates at right angles, is called the *Axis* of the parabola.

VI.

A third proportional to an absciss of a diameter of a parabola, and the corresponding ordinate, is called the *Parameter*, or *Latus Rectum* of the diameter. The parameter of the axis is frequently called the *Principal Parameter*, or *Latus Rectum*.

PROP. I.

BOOK
III.

If each of two diameters of a parabola meet a straight line, and if each of these straight lines cut, or one of them cut and the other touch the parabola, and if these two straight lines be parallel; then the segment of the one diameter, between its vertex and the line which it meets, will be to the segment of the other between its vertex and the line which it meets, as the square of the line which meets the first mentioned diameter if a tangent, or the rectangle under its segments if a secant, to the square of the line which meets the other diameter if a tangent, or the rectangle under its segments if a secant.

Suppose DI , GK to be two diameters of a parabola, and let D be the vertex of the one, and G the vertex of the other. Let LP , MN be two parallel straight lines, and let DI meet LP in L , and GK meet MN in M , and let LP , MN either both cut, or one of them cut and the other touch the parabola; then DL is to GM as the square of LP , if a tangent, or the rectangle under its segments if a secant, to the square of MN , if a tangent, or the rectangle under its segments if a secant.

Fig. 80.

For let $FGDC$ be the parabola as formed in the cone, and DI , GK the diameters mentioned above. Let the parabola cut the plane of the base in the straight line $FKIC$, and let the vertical plane cut it in BE , VB being the side along which the vertical plane touches the cone. Then DI , GK are parallel to VB , by the first Definition. Through the parallels VB , DI let a plane pass, and let it cut the cone in the side $VD A$, and the base in BIA . Through the parallels VB , GK let a plane pass, and let it cut the cone in the side $VG H$, and the base in BKH . Then (4. vi. and 16. v.)

Fig. 79.

$$DI : VB :: AI : AB, \text{ and}$$

$$VB : GK :: HB : HK.$$

H

Hence

BOOK Hence (I. vi.) $DI : VB :: AI \times IB : AB \times IB$
 III. and $VB : GK :: HB \times KB : HK \times KB$.

But (35. iii.) $AI \times IB$ is equal to $FI \times IC$, and $HK \times KB$ is equal to $FK \times KC$; and, by the seventh Lemma, $AB \times IB$ is equal to $HB \times KB$. Consequently, by the above and substitution, we have the two following ranks of magnitudes proportionals, taken two and two in the same order,

$$DI : VB : GK$$

$FI \times IC : AB \times IB : FK \times KC$; and therefore
 (22. v.) $DI : GK :: FI \times IC : FK \times KC$.

Let the straight line $FKIC$ have the same situation in the parabola in Fig. 80. as in Fig. 79. and first suppose LP, MN to be parallel to the base of the cone, or to FC . Then, by Prop. XVI. Book I. DL is to DI as the square of LP , if a tangent, or the rectangle under its segments, if a secant, to $FI \times IC$; and, by the above, DI is to GK as $FI \times IC$ to $FK \times KC$; and again, by Prop. XVI. Book I. GK is to GM as $FK \times KC$ to the square of MN if a tangent, or the rectangle under its segments if a secant. We have therefore

$$DL : DI : GK : GM$$

$$\left. \begin{array}{l} \text{t. } LP^2 \\ \text{or} \\ \text{f. } LP^r \end{array} \right\} : FI \times IC : FK \times KC : \left\{ \begin{array}{l} \text{t. } MN^2 \\ \text{or} \\ \text{f. } MN^r \end{array} \right.$$

Consequently (22. v.) DL is to GM as the square of LP if a tangent, or the rectangle under its segments if a secant, to the square of MN if a tangent, or the rectangle under its segments if a secant. Lastly, let LP, MN not be parallel to the base of the cone, but let LS, MR be parallel to the base, and let them touch or cut either of the conical superficies. Then, by the above, DL is to GM as the square of LS if a tangent, or the rectangle under its segments if a secant, to the square of MR if a tangent, or the rectangle under its segments if

if a secant. But, by Prop. XII. Book I. the square of LS if a tangent, or the rectangle under its segments if a secant, is to the square of MR if a tangent, or the rectangle under its segments if a secant, as the square of LP if a tangent, or the rectangle under its segments if a secant, to the square of MN if a tangent, or the rectangle under its segments if a secant. Consequently (II. v.) DL is to GM as the square of LP if a tangent, or the rectangle under its segments if a secant, to the square of MN if a tangent, or the rectangle under its segments if a secant.

BOOK
III.

PROP. II.

A diameter of a conic section bisects any straight line it meets in the section parallel to a tangent passing through its vertex; and ordinates to a diameter, and a tangent passing through its vertex, are parallel to one another.

In the ellipse and hyperbola this has been proved, according to Cor. 3. to Prop. III. Book II. In the parabola ABC let the diameter BG cut the straight line AC in the point G , and let AC be parallel to DE touching the parabola in B , the vertex of the diameter BG ; the straight line AC is bisected in G . On the contrary, any straight line in the parabola bisected by BG is parallel to AC , or the tangent DE . Fig. 81.

Part I. Through A, C let AD, CE be drawn parallel to BG , and let AD meet the tangent in D , and CE meet it in E . Then, by Cor. 1. Def. I. AD, CE are diameters, and, by Prop. I. $AD : CE :: DB^2 : BE^2$; and (34. i.) as AD, CE are equal, it follows that DB, BE are equal to one another. Consequently (34. i.) AG is equal to GC .

Part II. If it be possible, let the straight line HR in the parabola ABC be bisected by the diameter BG , and not be parallel to AC or DE .

H 2

Through

BOOK
III.

Through R draw RM parallel to the diameter BG , and through H draw HL parallel to AC or DE , and let it meet BG in K , the curve again in L , and RM in M . Let BG meet HR in N . Then as HR is bisected in N , and as KN , MR are parallel, (2. vi.) $HN : NR :: HK : KM$, and HK is equal to KM . But, by Part I. HK is equal to KL , and therefore KL is equal to KM ; which is absurd. Consequently no straight line in the parabola, unless it be parallel to DE , or to the ordinate AC , can be bisected by the diameter BG . Ordinates to the diameter BG must therefore be parallel to one another, and to the tangent DE , passing through the vertex.

Cor. 1. From hence, and Prop. III. Book II. it is evident, that if a straight line be an ordinate to a diameter, any straight line in the section, or opposite section, and parallel to it, will be an ordinate to the same diameter.

Cor. 2. From this Proposition it is evident, that if a straight line bisect two parallel lines in a conic section, it will be a diameter.

Cor. 3. From the above a method of finding a diameter of a given parabola is evident. For two parallel straight lines being drawn in the parabola, a straight line bisecting them, and any straight line parallel to it, will be a diameter.

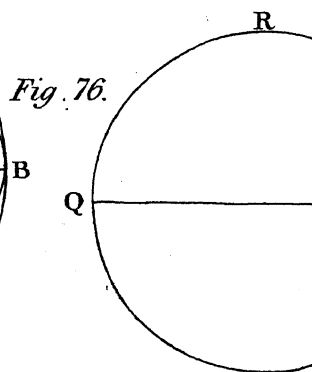
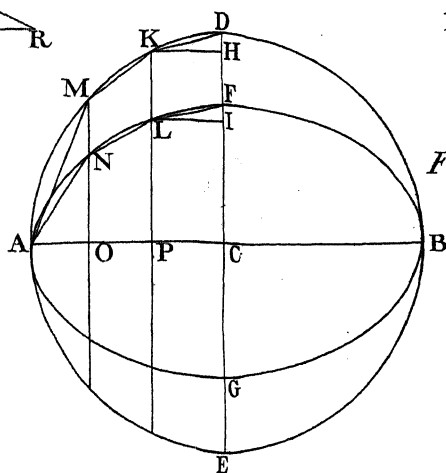
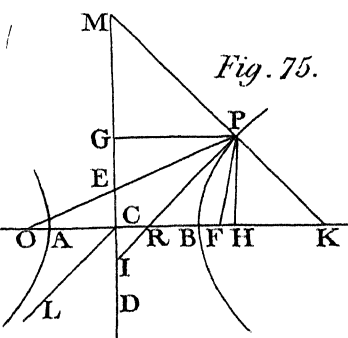
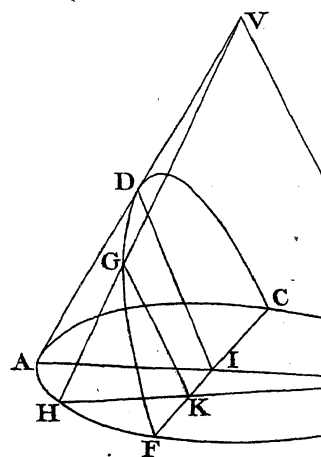
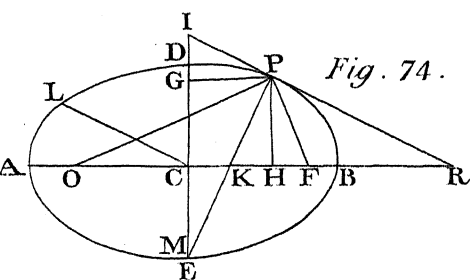
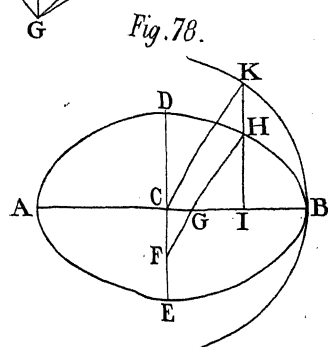
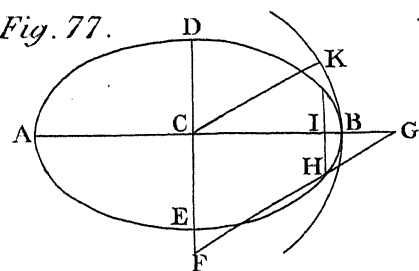
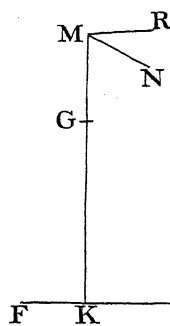
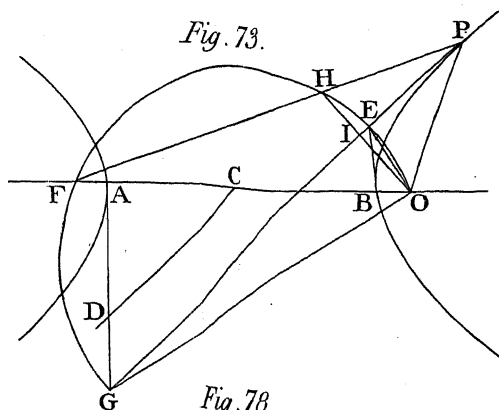
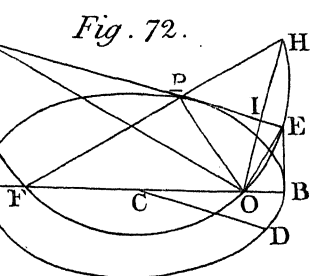
Cor. 4. The method of finding the axis of a parabola is also evident from the above. For, having found a diameter of the parabola, by the preceding Cor. let a straight line at right angles to it be drawn within the parabola, and limited both ways by the curve. Then the diameter bisecting this straight line will be the axis; for, being parallel to the diameter first found, it will (29. i.) bisect the straight line in the section at right angles.

PROP.

101

OK

8r.



100

BO
I

PROP. III.

BOOK
III.

The abscisses of a diameter of a parabola are to one another as the squares of the corresponding ordinates; and the square of an ordinate to a diameter of a parabola is equal to the rectangle under the parameter of the diameter, and the absciss corresponding to the ordinate.

Part I. Let BG be a diameter of the parabola ABC , and let AG , HK be ordinates to it, and let them meet it in the points G , K , and let B be the vertex of the diameter; the absciss BG is to the absciss BK as the square of the ordinate AG to the square of the ordinate HK .

Fig. 81.

For, as BG is parallel to a side of the cone, in which the section was formed, and as AG , HK , by Prop. II. are parallel, and as they would be bisected in G , K if limited by the curve, this part is evident from Prop. XVI. Book I. *

Part II. The rest remaining as above, let the straight line P be a third proportional to the absciss BG and the corresponding ordinate AG , and consequently the parameter to the diameter BG , according to the sixth Definition; the square of the ordinate HK is equal to the rectangle under P and the absciss BK .

For, by the preceding part, $BG : BK :: AG^2 : HK^2$, and therefore (I. vi.) $P \times BG : P \times BK :: AG^2 : HK^2$. But (17. vi.) $P \times BG$ is equal to AG^2 , and consequently (14. v.) $P \times BK$ is equal to HK^2 .

Cor. I. If a straight line touching a parabola meet a diameter, the square of the segment, between the point of contact and the point of concurrence, will be equal to

* From this and the second part of Prop. II. writers on projectiles prove, that, if the resistance of the air have no perceptible effect, a projectile must move in the curve of a parabola.

BOOK
III.

the rectangle under the segment of the diameter, between its vertex and the point of concurrence, and the parameter of the diameter, to whose ordinates the tangent is parallel. For let DB , touching the parabola in B , meet the diameter AD in D , and let AG parallel to DB meet the diameter BG in G , and let P be the parameter of BG . Then (34. i.) AG is equal to DB , and AD is equal to BG , and, by Prop. II. DB is parallel to the ordinates of BG ; and as, by the above, $BG \times P$ is equal to AG^2 , BD^2 is equal to $AD \times P$.

Cor. 2. If a straight line cutting a parabola meet a diameter, the rectangle under its segments, between the point of concurrence and the curve, will be equal to the rectangle under the segment of the diameter between its vertex and the point of concurrence, and the parameter of the diameter, to whose ordinates the secant is parallel. For, the rest remaining as in the preceding Cor. let the tangent BD be parallel to the straight line HM , cutting the parabola in H , L , and meeting the diameter RM in M . Then, by Prop. I. $AD : RM :: DB^2 : HM \times ML$; and therefore (1. vi.) $AD \times P : RM \times P :: DB^2 : HM \times ML$. Consequently, by the preceding Cor. (and 14. v.) $RM \times P$ is equal to $HM \times ML$.

SCHOLIUM.

On account of the equality of the square of HK to the rectangle under P and BK , Apollonius called the section a parabola.

From the property demonstrated above the parabola is frequently denoted by an algebraical equation, in the following manner. Put the parameter of the diameter $BK = p$, the absciss $BK = x$, and the ordinate $HK = y$. Then, from the above, $x : y :: y : p$, and $p x = y^2$.

PROP.

PROP. IV.

If each of two straight lines meeting one another touch or cut, or one of them touch and the other cut, a parabola, the square of the first of the two if a tangent, or the rectangle under its segments if a secant, will be to the square of the second if a tangent, or the rectangle under its segments if a secant, as the parameter of the diameter to whose ordinates the first is parallel, to the parameter of the diameter to whose ordinates the second is parallel.

For, first let the straight lines $A E$, $C E$, meeting one another in E , touch the parabola in A and C , and let $E B$ be a diameter passing through E . Then, by Cor. 1. Prop. III. the square of $A E$ is equal to the rectangle under $B E$, and the parameter of the diameter to whose ordinates $A E$ is parallel; and the square of $C E$ is equal to the rectangle under $B E$, and the parameter of the diameter to whose ordinates $C E$ is parallel. Consequently (I. vi.) the square of $A E$ is to the square of $C E$, as the parameter of the diameter to whose ordinates $A E$ is parallel to the parameter of the diameter to whose ordinates $C E$ is parallel. Fig. 82.

Next, let the straight line $G A$, touching the parabola in A , meet in G the straight line $G K$, which cuts the parabola in F , K , and let $G D$ be a diameter passing through G . Then, by Cor. 1. Prop. III. the square of $A G$ is equal to the rectangle under $D G$, and the parameter of the diameter to whose ordinates $A G$ is parallel; and, by Cor. 2. Prop. III. the rectangle $K G F$ is equal to the rectangle under $D G$, and the parameter of the diameter to whose ordinates $G K$ is parallel. Consequently (I. vi.) the square of $A G$ is to the rectangle $K G F$ as the parameter of the diameter to whose ordi-

BOOK
III.

names AG is parallel, to the parameter of the diameter to whose ordinates GK is parallel.

Lastly, if the straight line GK , cutting the parabola in P , K , meet in the point G the straight line GL , which cuts the parabola in the points H , L , then it may be proved in the same way, by means of Cor. 2. Prop. III. that the rectangle GKF is to the rectangle LGH as the parameter of the diameter to whose ordinates GK is parallel, to the parameter of the diameter to whose ordinates GL is parallel.

PROP. V.

If a straight line touching a parabola meet a diameter, and an ordinate to the diameter pass through the point of contact, the segment of the diameter, between its vertex and the tangent, will be equal to its absciss, between its vertex and the ordinate.

FIG. 23.

Let the straight line AE , touching the parabola ABC in the point A , meet the diameter BD in the point E , and through the point of contact A let the ordinate AD to BD pass, and meet BD in D ; the segment BE between the vertex and the tangent is equal to the absciss BD between the vertex and the ordinate.

For produce AD till it meet the curve in C , and draw CF parallel to ED , and let it meet AE in F . Then, by the third Definition, AC is bisected in D , and, by Cor. 1. to the first Definition, CF is a diameter; and, by Prop. I. $BE : CF :: AE^2 : AF^2$. But CF , DE being parallel, (2. vi.) $AD : DC :: AE : EF$, and AC being bisected in D , AF is bisected in E , and for the same reasons DE is half of CF . Consequently AF^2 is equal to four times AE^2 (4. ii.) and, therefore, CF is equal to four times BE ; and as CF is double of DE , BE is equal to BD .

Cor.

Cor. If AG , BD be any two diameters of the parabola ABC , and if AD be an ordinate to BD , and BG be an ordinate to AG , the abscissæ AG , BD will be equal. For let AE touch the parabola in A , and meet the diameter BD in E . Then, by *Cor. I.* to the first Definition, AG , EB are parallel, and, by *Prop. II.* AE , GB are parallel. Consequently (34. i.) AG is equal to EB , and therefore, as by the above EB , BD are equal, AG is equal to BD .

PROP. VI.

If two straight lines touching a conic section, or opposite hyperbolas, meet one another, the diameter bisecting the line joining the points of contact will pass through the point of concurrence.

In the ellipse, hyperbola, or opposite hyperbolas, this has been proved in *Prop. VIII.* Book II. In the parabola ABC let the two straight lines BA , EC touch the section in the points A , C , and meet one another in E , and let the diameter BD bisect AC , the straight line joining the points of contact in D ; the diameter BD will pass through E . Fig. 83.

For, as AC is bisected by the diameter BD , it is a double ordinate to BD ; and therefore, by *Prop. V.* if BD be produced and meet the tangents, its segment between B the vertex and the tangent AE will be equal to its abscissæ BD ; and its segment between B and the tangent CE will also be equal to BD . The diameter BD will therefore meet both the tangents AE , CE in the same point, and consequently will pass through E , the point of concurrence.

Cor. I. If two straight lines touching a conic section, or opposite hyperbolas, meet one another, a straight line passing through the point of concurrence, and bisecting

BOOK III. ing the line joining the points of contact, will be a diameter. The truth of this is evident from Cor. Prop. VIII. Book II. and the above.

Cor. 2. From the above it is evident, that if $A C$ be a double ordinate to $B D$, a diameter of any conic section $A B C$, and if $A E$, touching the section in A , meet the diameter in E , then if the straight line $E C$ be drawn, it will touch the section in C .

DEFINITIONS.

VII.

Fig. 84.

If from B , the vertex of the axis $A B$ of the parabola $P B M$, a segment $B F$ be taken in the axis equal to one fourth of the parameter of the axis, the point F is called the *Focus*, or *Umbilicus*, of the parabola.

Cor. The double ordinate $T S$, drawn through F the focus of any conic section, is equal to the parameter of the axis passing through the focus. This has been proved in the ellipse and hyperbola in Cor. 4. to the eleventh Definition in Book II. In the parabola the square of $T F$ is equal to the rectangle under $B F$, and four times $B F$, by this Def. and Prop. III. and therefore $4 T F^2$ is equal to $4 B F \times 4 B F$. But (4. ii.) $4 T F^2$ is equal to $T S^2$, and consequently $T S$ is equal to $4 B F$, and therefore, by this Def. equal to the parameter of the axis $A B$.

VIII.

As in the ellipse and hyperbola, so in the parabola, the straight line touching the section in T , the extremity of the double ordinate drawn as above, is called the *Focal Tangent* to the parabola.

IX.

The transverse axis of an ellipse or hyperbola, and the axis of a parabola, is sometimes called the *Focal Axis* of the section.

X. If

X.

BOOK
III.

If the focal tangent $τ G$, belonging to the focus F in any conic section PBM , meet the focal axis AB in X , the straight line XY at right angles to AB is called a *Directrix* of the section. And, if in the ellipse or hyperbola O be the other focus, and the focal tangent belonging to O meet the focal axis AB in K , the straight line KL at right angles to AB is also called a directrix of the ellipse or hyperbola.

Fig. 84.
85.
86.

Cor. 1. As in the ellipse and hyperbola the foci F, O are equally distant from C the center, it is evident from the above, and Prop. VII. Book II. that the directrices XY, KL are equally distant from the center.

Cor. 2. In the parabola, the focus F and the directrix XY are equally distant from B , the vertex of the axis, by the above and Prop. V.

PROP. VII.

If a tangent passing through the vertex of the focal axis of a conic section meet a focal tangent, its segment between the point of contact and point of concurrence will be equal to the segment of the axis between the point of contact and the focus to which the focal tangent belongs.

Let the tangent BG , passing through B the vertex of the focal axis AB , of any conic section PBM , meet in the point G the focal tangent $τ G$ belonging to the focus F ; the segment GB is equal to the segment FB . And in the ellipse and hyperbola the tangent AH , passing through A the other vertex of the focal axis, and meeting the focal tangent $τ G$ in H , is equal to the segment AF .

Fig. 84.
85.
86.

As far as this Proposition relates to the ellipse, or hyperbola, it has been proved in Prop. XII. Book II.

In

BOOK
III.

In the parabola, every thing remaining as in the seventh Definition and its Cor. TF is double of FB , and therefore, by Cor. 2. to the tenth Definition, TF is equal to FX . Again, by Prop. II. TF , GB are parallel, and therefore (4. vi.) $TF : FX :: GB : BX$, and GB is equal to BX , and consequently equal to FB .

PROP. VIII.

A straight line drawn from any point in the curve of a conic section to a focus is to a straight line drawn from the same point perpendicular to the directrix nearest this focus, as the segment of the axis between the same focus and the nearest vertex, to the segment between this vertex and the directrix: and, in the ellipse and hyperbola, a straight line drawn from the same point in the curve to the other focus is to a straight line drawn from the same point perpendicular to the other directrix in the same ratio.

Fig. 84.
85.
86.

A straight line PF , drawn from any point P in the curve of the conic section PBM to the focus F , is to PY perpendicular to XY , the directrix nearest to F , as the segment FB , of the focal axis between F and the vertex B , to the segment BX of the same axis between B the vertex and the directrix: and, in the ellipse and hyperbola, PO drawn to the other focus O is to PQ drawn perpendicular to the other directrix KL in the same ratio.

For, the rest remaining as in the preceding Prop. and Def. X. through P draw PM an ordinate to the axis AB , and let it meet the curve again in M , the axis AB in R , and the focal tangent TG in N . Then, by Prop. XIII. Book I. $TG^2 : TN^2 :: GB^2 : PN \times NM$. But, by Prop. II. NM , TF , GB are parallel, and therefore (10. and 22. vi.) $TG^2 : TN^2 :: FB^2 : PR^2$; and therefore

(11.

(II. v.) $FB^2 : FR^2 :: GB^2 : PN \times NM$. Consequently, as by Prop. VII. FB is equal to GB , FB^2 is equal to GB^2 , and (14. v.) FR^2 is equal to $PN \times NM$; and therefore (6. ii. and 47. i.) RN^2 is equal to PF^2 , and RN is equal to PF . But (4. vi.) $RN : RX :: GB : BX$, and therefore as (34. i.) PY is equal to RX , and, by Prop. VII. FB is equal to GB , $PF : PY :: FB : BX$.

BOOK
III.

Again, in the ellipse and hyperbola, the rest remaining as above, let c be the center, and let nv perpendicular to the directrix KL meet KL in v . Then (34. i.) nv , pa are equal, and vk is equal to nr , and consequently equal to pf . Let cd the semiconjugate axis be produced till it meet the focal tangent tg in i , and, by Cor. 2. Prop. XIII. Book II. ci will be equal to cb or ca . Also (4. vi.) $xc : ci :: xk : kl$; and therefore, as xc , ck are equal, kl is equal to ab the transverse axis. Consequently, by Cor. 1. Prop. XIII. Book II. lv is equal to po ; and as lv is parallel to gb , and vn to bx , $lv : vn :: gb : bx$. On account of the equals therefore, $po : pq :: fb : bx$.

Cor. 1. In the ellipse and hyperbola, (4. vi.) $ci : cx :: nr : rx$; and therefore on account of the equals $cb : cx :: pf : py$. But, by Prop. VII. Book II. $cb : cx :: cf : cb$; and therefore (II. v.) $cf : cb :: pf : py$. For the same reasons, or by the above, (and II. v.) $cf : cb :: fb : bx$.

Cor. 2. By the preceding Cor. in the ellipse pf is less than py , but in the hyperbola pf is greater than py . And, as by the above (and II. v.) $pf : py :: po : pq$, in the ellipse po is less than pa , but in the hyperbola po is greater than pa .

Cor. 3. A straight line drawn from any point in the curve of a parabola to the focus is equal to a straight line

Fig. 84.

BOOK III. line drawn from the same point perpendicular to the directrix. For $PF : PY :: FB : BX$, and FB is equal to BX .

SCHOLIUM.

Some writers on conic sections have chosen this property as the primary one for their treatises, and according to it have defined the sections in the following manner.

Fig. 84.
85.
86.

Let F be a point without the straight line xy , and whilst a straight line FP revolves about F as a center, let a point P so move in FP that FP may always be to PY , perpendicular to xy , in a given ratio. The curve described by the point P will be a conic section; and it will be a parabola, ellipse, or hyperbola, according as FP is equal to, less, or greater than PY .

PROP. IX.

If from any point in the curve of a parabola a straight line be drawn to the focus, and a straight line perpendicular to the directrix, the angle contained by these straight lines will be bisected by a tangent passing through the same point.

Fig. 89.

From the point P in the curve of the parabola PBR let the straight line PF be drawn to F the focus, and the straight line PD perpendicular to DX the directrix; the straight line PE , touching the parabola in P , bisects the angle FPD .

For let the tangent PE meet the axis AB in E , and let PA be an ordinate to AB . Then, by Prop. V. AE is equal to BE ; and as, by Cor. 2. to the tenth Definition, FB is equal to BX , AX is therefore equal to FE . But (34. i.) AX is equal to PD ; and, by Cor. 2. Prop. VIII. PD is equal to PF . Consequently PE ,
F E

FE are equal, and therefore (5. i.) the angle FPE is equal to the angle FEP . But as PD , AE are parallel, the angle FEP (29. i.) is equal to the angle DPE . Consequently the angle FPE is equal to the angle DPE , and therefore the angle FPD is bisected by the tangent PE .

Cor. If a straight line touching a parabola meet the axis, the segment of the axis between the point of concurrence and the focus is equal to the straight line drawn from the point of contact to the focus. This is evident from the above, for FE is equal to PF .

SCHOLIUM.

It is not certain when, or by whom, the name *Focus*, or *Umbilicus*, was first given to a point in a conic section. Neither of the two occurs in the Treatise of Apollonius, or in the writings of Archimedes, who occasionally mentions properties of the sections. The points themselves, however, in the ellipse and hyperbola, were well known to Apollonius. He calls them *puncta ex applicatione facta*; and he demonstrates the most important properties of lines related to them. He does not mention the focus of the parabola.

It is highly probable, from analogy of general principles, and from the history of this branch of science as far as it has been traced, that optical pursuits first suggested each of the two names. For, in opticks, if a ray of light fall upon a plane surface, the angle of incidence is equal to the angle of reflexion; and if a ray of light fall upon a curve, and a straight line touch the curve in the same point, the angle contained by the incident ray and tangent will be equal to the angle contained by the reflected ray and tangent, as a tangent is the direction of a curve in the point of contact.

These

BOOK
III.

These truths being premised, suppose a conic section RBP to revolve about AB , the focal axis, and that a concave speculum is formed by the curve by this revolution. Let the straight line TPE touch any one of the sections in the point P . In the ellipse and hyperbola let F, o be the foci, and let F be the focus of the parabola.

Fig. 89.
90.
91.

1. Let KP be a ray of light, whose direction, in a straight line, passes through o the focus opposite to F , within the hyperbolic speculum RBP . Let it fall upon the speculum in the point P , and draw PF , and produce KP to o . Then, as by Prop. XV. Book II. the angle FPE is equal to the angle oPE , and consequently (15. i.) equal to the angle KPT , the straight line PF will represent the ray of light after reflexion. For, if a perpendicular to TE be drawn from P , the angle contained by it and KP will be equal to the angle contained by it and PF ; the angle of incidence being equal to the angle of reflexion. Hence if any number of rays fall on a concave hyperbolic speculum, and be converging to the focus in the opposite hyperbola, they will be reflected to the focus within the speculum.

Fig. 90.

2. Let $ARDBP$ be a concave elliptic speculum. Let a ray of light proceeding from the focus o fall upon the speculum at P , and let PF be drawn. Then will PF represent the ray after reflexion, for the same reasons as above, as the angles OPT, FPE are equal, by Cor. 1. to Prop. XV. Book II. Hence it is evident, that if any number of rays proceed from one focus of an elliptic concave speculum, they will be reflected into the other. As rays of heat are subject to the same laws with rays of light, as to incidence and reflexion, if a fire or any heated body be placed at o , within the concave elliptic speculum, the whole of the heat after reflexion will meet at F . Perhaps attention

to

to this property might be of considerable use in fitting
up fire-places, reverberating furnaces, &c. BOOK
III.

3. Let κP be a ray of light parallel to AB the axis of the parabola, by whose revolution the parabolic speculum is generated, and let PF be drawn. Then will PF represent the ray after reflexion. For, κP being produced to D , the angle FPD will be equal to the angle DPE , by the Proposition preceding this Scholium, and therefore (15. i.) the angle KPT will be equal to the angle FPE . Consequently, as above, PF is the ray after reflexion. Hence if any number of rays, parallel to the axis, fall upon a concave parabolic speculum they will all after reflexion meet in the focus. Fig. 89.

The very considerable magnifying powers which reflecting telescopes are capable of, with parabolic specula, are to be attributed to this property. For a celestial body being at an immense distance, the rays which issue from it upon the parabolic speculum are, as to sense, parallel to the axis; and, being all reflected to the focus, a distinct and vivid image of the body is produced, provided the composition of the metal be good and the parabolic figure just.

PROP. X.

A straight line drawn from the focus of a parabola, perpendicular to a tangent, is a mean proportional between the straight line drawn from the point of contact to the focus, and the segment of the axis between the focus and the vertex of the axis.

From F the focus of the parabola PBR let the straight line FG be drawn perpendicular to the straight line PE , touching the parabola in P , and draw PF ; FG is a mean proportional between PF and FB , the segment of Fig. 89.

BOOK of the axis AB , between F and B the vertex of the
 III. axis.

For let the tangent FE meet the axis in E , and draw BG , and let PA be an ordinate to the axis. Then, by Cor. Prop. IX. PF , FE are equal, and therefore (5. i.) the angles FPE , FEP are equal. Consequently in the triangles FGP , FGE , as the angles at G are right angles, PG is (26. i.) equal to GE , and the angles FGP , EGP are equal. Consequently, as, by Prop. V. AB is equal to BE , $PG : GE :: AB : BE$, and (2. vi.) GB is parallel to the ordinate PA , and therefore GBF is a right angle. The triangles FGP , GFB are therefore equiangular, and (4. vi.) $PF : FG :: FG : FB$.

The above Prop. is Lemma XIV. Lib. I. of the Principia.

Cor. 1. Hence (Cor. 2. 20. vi.) $PF^2 : FG^2 :: PF : FB$.

Cor. 2. The concourse of any tangent FE with a straight line FG , drawn from the focus of the parabola perpendicular to the tangent, is in the straight line BG , which touches the parabola in the vertex of the axis. For, by the above, GB is parallel to the ordinate PA , and therefore the Cor. is evident by Prop. II.

Cor. 3. If a straight line touch a parabola, and cut a straight line drawn from the focus to the directrix at right angles, it will bisect it. For, the rest remaining as above, let FG produced meet the directrix in D . Then as GB , DX are perpendicular to the axis, they are parallel, and (2. vi.) $FB : BX :: FG : GD$, and as, by Cor. 2. Def. X. FB is equal to BX , FG is equal to GD .

SCHOLIUM.

If a straight line pass through a point moving in the curve of a conic section, and always touch the section,
 and

and if a straight line revolve about a focus of the section as a center, and be always perpendicular to the moving tangent, the magnitude of the perpendicular will be less varied in the hyperbola than in the parabola, but it will be more varied in the ellipse than in the parabola.

For let the point P be supposed to move in the curve BP of the conic section PBR , and let the straight line PE accompany it in its motion, and always touch the section. Let the straight line FG revolve about F , a focus of the section, and let it be always perpendicular to the tangent PE .

Fig. 89.
90.
91.

In the ellipse and hyperbola let c be the center, CD the semiconjugate axis, and CH the semidiameter parallel to the tangent PE . Then, by Prop. XIX. Book II. $FG : FP :: CD : CH$, and therefore, by Lemma V. $FG^2 : FP^2 :: CD^2 : CH^2$. But o being the other focus, and PO being drawn, by Prop. XVIII. Book II. $CH^2 = FP \times PO$, and therefore $FG^2 : FP^2 :: CD^2 : FP \times PO$, and $FG^2 = \frac{FP^2 \times CD^2}{FP \times PO} = \frac{FP \times CD^2}{PO}$. For the

Fig. 90.
91.

same reasons if p denote another position of the moving point, Fg the perpendicular at that position, and Fp , po straight lines drawn from p to the foci, then $Fg^2 = \frac{Fp \times CD^2}{po}$. Consequently $FG^2 : Fg^2 :: \frac{FP \times CD^2}{PO} : \frac{Fp \times CD^2}{po}$.

$$\frac{FG^2}{Fg^2} :: \frac{FP}{Fp} : \frac{PO}{po}.$$

If in the parabola p denote another position of the moving point, Fg the perpendicular at that position, and pF a straight line drawn to the focus, then, by the Proposition preceding this Scholium, (and 17. vi.) $Fg^2 = Fp \times FB$, and $FG^2 : Fg^2 :: FP \times FB : Fp \times FB$.

$$FG^2 : Fg^2 :: \frac{FP}{Fp} : \frac{FB}{FB}.$$

BOOK
III.

In the hyperbola and ellipse, therefore, the square of the perpendicular FG varies as the value of the fraction $\frac{FP}{PO}$ varies, and in the parabola it varies as $\frac{FP}{FB}$ varies.

But if the numerator and denominator of a fraction be each variable, then if they always increase or decrease in the same proportion, the value of the fraction will be always the same. Thus if the fraction be $\frac{a}{r}$, and if while a varies and becomes q , r varies and becomes r , and if it be $a : r :: q : r$, then $\frac{a}{r} = \frac{q}{r}$, by converting the proportion into an equation. From hence it is also evident, that the more nearly the numerator and denominator increase or decrease in the same proportion, the less will the value of the fraction be varied; but, on the contrary, the more they differ from a proportional increase or decrease, the more will the value of the fraction be varied. Now in the hyperbola the difference between FP , PO is, in every situation of P , equal to AB the transverse axis, and therefore, in every instant they vary, they receive an equal increase or diminution; but however, in the parabola, FP may increase or diminish, FB remains constant. In the hyperbola therefore the value of the fraction $\frac{FP}{PO}$ varies less than the value of the fraction $\frac{FP}{FB}$ in the parabola. Again, in the ellipse the sum of FP , PO is equal to AB the transverse axis, and therefore if either of them increase, the other will diminish; and consequently in varying they will differ more from a proportional increase at the same time, or decrease at the same time, than FP , FB in the parabola. In the ellipse

lipse therefore the value of the fraction $\frac{FP}{PO}$ varies more BOOK
III.

than the value of the fraction $\frac{FP}{FB}$ in the parabola.

In the hyperbola therefore the square of FG varies less, but in the ellipse it varies more, than it varies in the parabola; and consequently FG varies less in the hyperbola, but more in the ellipse, than it varies in the parabola. See the Principia, Cor. 6. Prop. XVI. Lib. I.

PROP. XI.

If from a point in which a straight line touches a parabola two straight lines be drawn to the axis, one of them an ordinate to it, and the other at right angles to the tangent, the segment of the axis intercepted between them will be equal to half the parameter of the axis: and if a straight line touching a parabola meet the axis, and a straight line, at right angles to the tangent, be drawn from the point of contact to the axis, the segment of the axis intercepted between them will be equal to half the parameter of the diameter passing through the point of contact.

Part I. From the point P , in which the straight line PE touches the parabola PBR , let the two straight lines PA , PH be drawn to AB the axis, one of them PA an ordinate to it, and the other PH at right angles to the tangent PE ; the segment HA of the axis, intercepted between them, is equal to half the parameter of the axis. Fig. 89.

For let the tangent PE meet the axis in E , and then as PA is an ordinate to the axis, it is at right angles to HE ; and (Cor. 8. vi.) the square of PA is equal to the rectangle under EA , AH . But, by Prop. V. EB , EA are equal, and, by Prop. III. the square of PA is equal

BOOK to the rectangle under BA and the parameter of the
III. axis. Consequently the rectangle under EA , AH is
 equal to the rectangle under BA and the parameter of
 the axis, and therefore (16. vi.) EA is to BA as the pa-
 rameter of the axis to AH ; and as BA is half of EA ,
 AH must be half of the parameter of the axis.

Part II. Let the straight line PE touching the para-
 bola PBR in P meet the axis AB in E , and let the
 straight line PH , at right angles to PE , meet the axis
 AB in H ; the segment EH , intercepted between PE ,
 PH , is equal to half the parameter of the diameter PK ,
 passing through the point of contact.

For, the rest remaining as in the preceding part,
 the square of PE (Cor. 8. vi.) is equal to the rectangle
 under HE , EA ; and, by Cor. 1. Prop. III. the square
 of PE is equal to the rectangle under EB and the para-
 meter of PK . Consequently the rectangle under HE ,
 EA is equal to the rectangle under EB and the para-
 meter of PK , and therefore (16. vi.) EA is to EB as
 the parameter of PK to HE ; and as EB is half of EA ,
 HE must be equal to the half of the parameter of PK .

Cor. 1. The parameter of the axis is less than the
 parameter of any other diameter.

Cor. 2. A straight line drawn from any point in the
 curve of a parabola to the focus is equal to a fourth
 part of the parameter of the diameter passing through
 the same point. If the point be the vertex of the axis,
 this is evident from the seventh Definition; but for
 any other point P let every thing remain as in this
 Prop. and draw PF to the focus F . Then, by Cor.
 Prop. IX. FP , FE are equal; and as EPH is a right
 angle, if with F as a center, and FP as a distance, a cir-
 cle be described, it will pass (31. iii.) through E and H .
 Consequently EF , FP , PH are equal, and therefore, by
 Part II. of this Prop. each of them is equal to a fourth
 part

part of the parameter of PK . This is Lemma XIII. BOOK
III.
Lib. I. of the Principia.

Cor. 3. The distance of the vertex of any diameter of a parabola from the directrix is equal to a fourth part of the parameter of the diameter. This is evident from the preceding *Cor.* and *Cor. 3. Prop. VIII.*

PROP. XII.

Any straight line drawn through the focus of a parabola, and terminated both ways by the curve, is equal to the parameter of the diameter to which it is a double ordinate.

Let the straight line AH , passing through F the focus of the parabola ABC , meet the curve in A and H , and be a double ordinate to the diameter DE ; AH is equal to the parameter of DE . Fig. 87.

If DE be the axis, the *Prop.* is the same as the *Cor.* to the seventh Definition. Let DE therefore not be the axis, and let it meet AH in E . Let DG touch the parabola in D the vertex of DE , and meet BF the axis in G , and draw DF . Then, by *Cor. 1.* to the first Definition, and *Prop. II.* GE is a parallelogram, and therefore (34. i.) DE, GF are equal. But, by *Cor. Prop. IX.* GF, DF are equal, and, by *Cor. 2. Prop. XI.* DF is equal to a fourth part of the parameter of DE . Consequently DE is equal to a fourth part of the parameter of DE , and therefore, by *Prop. III.* AE^2 is equal to $DE \times 4 DE$, and $4AE^2$ is equal to $4DE \times 4DE$. But (4. ii.) $4AE^2$ is equal to AH^2 , and therefore AH is equal to $4DE$, and consequently equal to the parameter of DE .

Cor. If a straight line in a parabola pass through the focus, and cut the diameter to which it is an ordinate, the absciss of the diameter will be equal to the distance

BOOK III.
 of its vertex from the focus, and also from the directrix. For, as above, DE is equal to DF , and therefore, by Cor. 3. Prop. VIII. the remaining part of this is evident.

DEFINITIONS.

XI.

Fig. 88. The superficies $ACLBIA$, bounded by the straight line AC and the curve $AIBLC$ of a parabola, is called an *interior parabolic segment*; and if the straight lines AE , CE , touching the parabola in A , C , meet one another in E , the superficies bounded by AE , CE , and the curve $AIBLC$, is called the *exterior parabolic segment*, corresponding to the interior segment first mentioned.

XII.

If the straight line RS , parallel to AC , touch the parabola in B , and meet in the points R , S the diameters AR , CS , the parallelogram $ARSC$ is said to be circumscribed about the interior parabolic segment $AIBLC$.

PROP. XIII.

In an interior parabolic segment a rectilineal figure may be inscribed, and a corresponding rectilineal figure may be inscribed in the exterior segment, so that the rectilineal figure in the interior segment shall be double of that inscribed in the exterior; and these rectilineal inscribed figures may be such, that each shall be less than the segment in which it is inscribed by a superficies less than any given superficies.

Fig. 88. Let $ACLBIA$ be an interior, and $E A I B L C E$ the corresponding exterior parabolic segment, as in the eleventh Definition; a rectilineal figure may be inscribed in the segment $ACLBIA$, and a corresponding recti-

rectilineal figure may be inscribed in the segment $E A I B L C E$, so that the rectilineal figure in the interior segment shall be double of that in the exterior; and these rectilineal figures may be such, that each shall be less than the segment in which it is inscribed by a superficies less than a given superficies O .

For bisect AC in D , and draw ED , and let it cut the curve in B . Let the straight line FG touch the parabola in B , and meet the tangent EA in F , and the tangent EC in G , and draw AB , CB . Then, by Cor. 1. Prop. VI. ED is a diameter of the parabola, and, by Prop. II. FG , AC are parallel; and, by Prop. V. DB , BE are equal. Consequently (2. vi.) AF , FE are equal, and (1. vi.) the triangle AFB is equal to the triangle EFB ; and therefore, as (1. vi.) the triangle ABD is equal to the triangle AEB , the triangle ABD is double of the triangle EFB . For the same reasons the triangle CBD is double of the triangle ECB ; and therefore the triangle ABC , inscribed in the interior parabolic segment, is double of the triangle EBG , inscribed in the exterior parabolic segment. Again, bisect AB in H , and draw FH , and let it meet the curve in I . Let the straight line MN touch the parabola in I , and meet the tangent AF in M , and the tangent FB in N , and draw AI , IB . Then, as above, it may be proved, that the triangle AIB , inscribed in the interior parabolic segment $AIBA$, is double of the triangle FMN , inscribed in the exterior parabolic segment $FAIBF$. Also bisect BC in K , and draw GK , and let it meet the curve in L . Let the straight line PQ touch the parabola in L , and meet the tangent CG in Q , and the tangent BG in P , and draw BL , LC . Then it may be also proved, as above, that the triangle BLC , inscribed in the interior parabolic segment $BLCB$, is double of the tri-

BOOK
III.

BOOK III. triangle $G P Q$, inscribed in the exterior parabolic segment $G B L C G$. Consequently the rectilineal figure $A I B L C$, inscribed in the interior parabolic segment first mentioned, is double of the rectilineal figure $E M N P Q$, inscribed in the corresponding exterior parabolic segment first mentioned. In the same manner the inscription in each segment may be continued, so that the whole figure inscribed in the interior segment shall be double the whole figure inscribed in the exterior segment.

The inscription may also be continued till the rectilineal figure shall be less than the segment in which it is inscribed by a superficies less than the superficies O . For in the interior segment the triangle $A B C$, being equal to half the triangle $E A C$, is greater than half the segment $A I B L C A$; and for the same reasons the triangle $A I B$ is greater than half the segment $A I B A$, and so of other remaining segments. Also in the exterior segment the triangle $E F G$, being half of the rectilineal figure $E A B C E$, is greater than half the segment $E A I B L C E$; and for the same reasons the triangle $F M N$ is greater than half the segment $F A I B F$, and so of other remaining segments. Consequently (I. x.) the inscription may be continued till each of the rectilineal figures shall be less than the segment in which it is inscribed by a superficies less than the given superficies O .

Cor. The tangent $P G$ being produced till it meet the diameters $A R$, $C S$ in R and S , the circumscribed parallelogram $R A C S$ is equal to the triangle $E A C$, and therefore equal to the interior and exterior parabolic segments taken together. For, by the above, $E F$, $F A$ are equal, and as $R A$, $E B$ are parallel, the triangles $E F B$, $A F R$ are equiangular, and (26. and 4. i.)

4. i.) therefore equal. For the same reasons the tri- BOOK
III.
angles $\angle EGB$, $\angle GCS$ are equal; and consequently the
Cor. is evident.

PROP. XIV.

An interior parabolic segment is double of the corresponding exterior parabolic segment; and the interior parabolic segment is to the circumscribed parallelogram as two to three.

Every thing remaining as in the preceding Prop. and Fig. 22, its Cor. the interior parabolic segment $ACLBIA$ is double the corresponding exterior parabolic segment $E A I B L C E$, and the interior segment is to the circumscribed parallelogram $RACS$ as two to three.

Part I. If the interior segment be not double of the exterior segment, it must either be greater or less than its double. First, let it be greater than the double of the exterior segment, and let the superficies o be equal to the excess of the interior segment above the double of the exterior. Let corresponding rectilineal figures be inscribed in the interior and exterior segments, as in the preceding Prop. and let them be such that the rectilineal figure $AIBLC$, inscribed in the interior segment, may be less than the segment in which it is inscribed by a superficies less than o , and let $EMNPAE$ be the corresponding rectilineal figure inscribed in the exterior segment. Then the rectilineal figure $AIBLC$ is greater than the double of the exterior segment, and therefore its half is greater than the exterior segment. But the rectilineal figure $EMNPAE$ is half of $AIBLC$, by Prop. XIII. and consequently the rectilineal figure $EMNPAE$ is greater than the parabolic segment in which it is inscribed: which is absurd. The interior segment therefore cannot be greater than the double of
the

BOOK
III.

the exterior segment. Secondly, let the interior segment be less than the double of the exterior segment, and consequently the exterior segment greater than half the interior segment. Let the excess of the exterior segment above half the interior be equal to the superficies o ; and let corresponding rectilineal figures be inscribed in each segment, as in the preceding Prop. so that the rectilineal figure $EMNPQ E$ inscribed in the exterior segment may be less than the segment in which it is inscribed by a superficies less than o . Then the rectilineal figure $EMNPQ E$ is greater than half the interior segment, and therefore its double is greater than the interior segment. Let $AIBLCA$, $EMNPQ E$ be the corresponding rectilineal figures inscribed in the two segments; and then, by Prop. XIII. as $AIBLCA$ is double of $EMNPQ E$, it is greater than the interior segment in which it is inscribed: which is absurd. The interior segment is therefore not less than the double of the exterior segment. Consequently as the interior segment is neither greater nor less than the double of the exterior, it is equal to the double of the exterior segment.

Part II. As, by Part I. the interior segment is double of the exterior, the interior segment is to the exterior as two to one, and therefore (18. v.) the interior segment is to the triangle EAC as two to three. Consequently, by Cor. Prop. XIII. the interior segment is to the circumscribed parallelogram $RACS$ as two to three*.

* Archimedes was the first who proved that a parabolic segment is equal to two thirds of the circumscribed parallelogram. Of this truth he gave two demonstrations. The first may be considered as mechanical, as it depends upon the primary properties of the lever. The second is strictly geometrical, and may be easily understood from the above; for he proves, that the triangle AED is equal to four times the

DEFINITIONS.

BOOK
III.

XIII.

Fig. 92.

If AMN be the vertical plane to the opposite hyperbolas EVF , GLH , as in the eighteenth Definition of the first Book, and cut the cone in the sides AM , AN , and if planes AK , AI , touching the cone in the sides AN , AM , cut the plane of the hyperbolas in the straight lines KQ , IR , the straight lines KQ , IR are called the *Asymptotes* of either of the hyperbolas, or of the opposite hyperbolas.

Cor. 1. As the vertical plane AMN is parallel to the plane of the hyperbolas EVF , GLH , the asymptotes KQ , IR (16. xi.) are parallel to AN , AM , sides of the cone.

Cor. 2. The asymptotes KQ , IR do not meet the curve of either of the opposite hyperbolas. For the planes AK , AI touch the opposite cones in the sides AN , AM , and as the asymptotes are parallel to these sides, they do not meet either of the opposite conical superficies. The *Cor.* is therefore evident.

Cor. 3. Any two straight lines TS , PO drawn in the planes AK , AI from the asymptotes to the sides AN , AM , and parallel to the base of the cone, are equal, and touch the conical superficies. For let NM , IK , NK , MI be the lines of common section of the base with the vertical plane, the plane of the hyperbolas,

the triangle AIB , each of these triangles being inscribed according to the conditions stated above. The remainder of his demonstration is then equivalent to this, (see Maclaurin's Algebra, §. 68.) that the sum of the infinite series $1 + \frac{x}{4} + \frac{x^2}{16} + \frac{x^3}{64} + \&c.$ can neither be greater or less than $\frac{4}{3}$, the triangle ABC being analogous to 1.

The above Proposition and the foregoing remarks being clearly comprehended, the 5th *Cor.* to Lemma XI. of Sir Isaac Newton's Doctrine of Prime and Ultimate Ratios will be easily understood.

and

BOOK
III.

and the planes AK , AI . Then, as AK , SN are parallel, and as ST , being parallel to the base, is parallel to NK , the straight line ST (34. i.) is equal to NK . For the same reasons OP is equal to MI . Also NK , MI are equal. For if NK , MI be parallel, then MK is a parallelogram, and (34. i.) NK is equal to MI . But if NK be not parallel to MI , let them meet in w ; and as they are in the tangent planes, they will touch the circle MDN , the base of the cone. The straight lines WN , WM (36. iii.) are therefore equal, and (2. vi.) $WN : NK :: WM : MI$. Consequently (14. v.) NK is equal to MI , and therefore ST , OP are equal, and as they are in the planes AK , AI , they touch the conical superficies.

XIV.

The angle ICK , or ACK , within which either of the opposite hyperbolas is situated, is called the *interior angle of the asymptotes*; and the angle RCI , or ACI , adjacent to it, is called the *exterior angle of the asymptotes*.

PROP. XV.

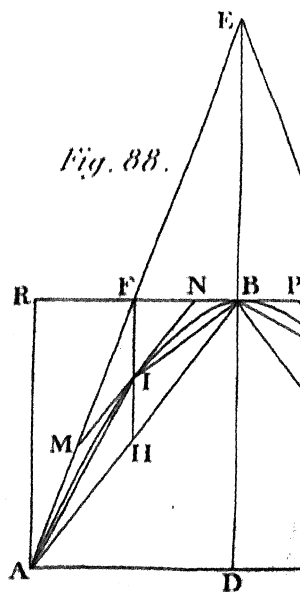
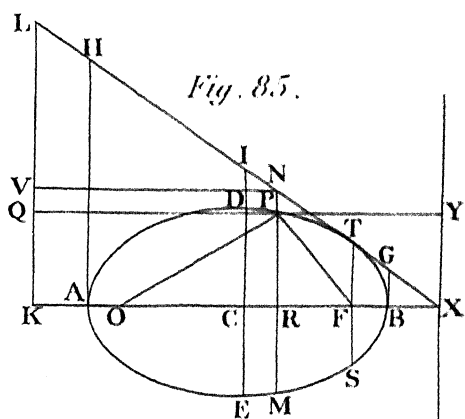
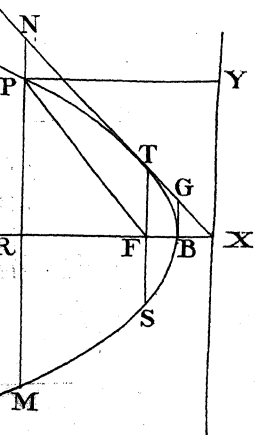
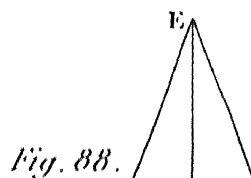
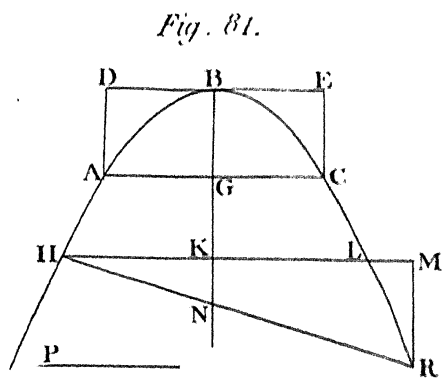
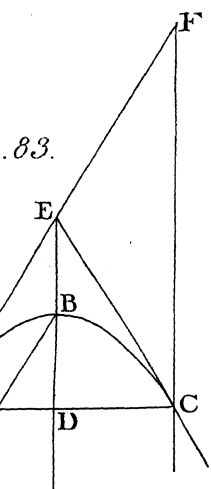
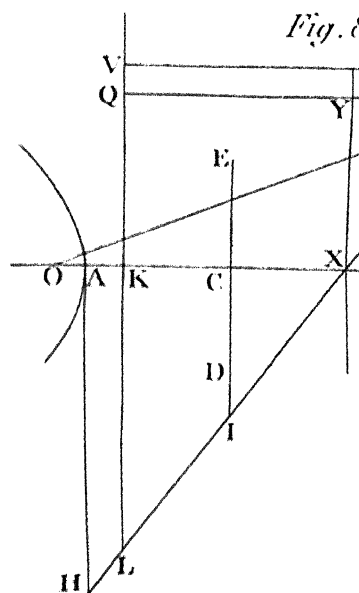
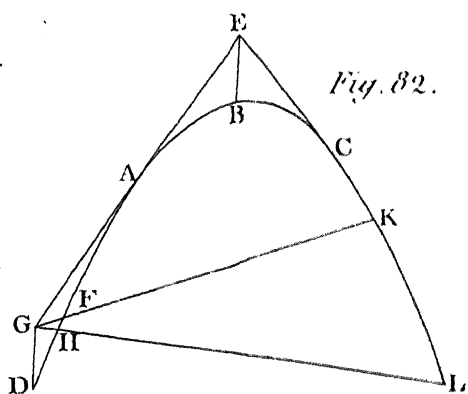
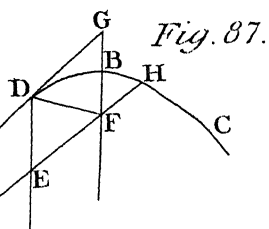
The point in which the asymptotes of an hyperbola cut one another is the center of the hyperbola; and any straight line passing through the center and falling within the interior angle of the asymptotes is a transverse diameter; but any straight line passing through the center and falling within the exterior angle of the asymptotes is a second diameter of the hyperbola.

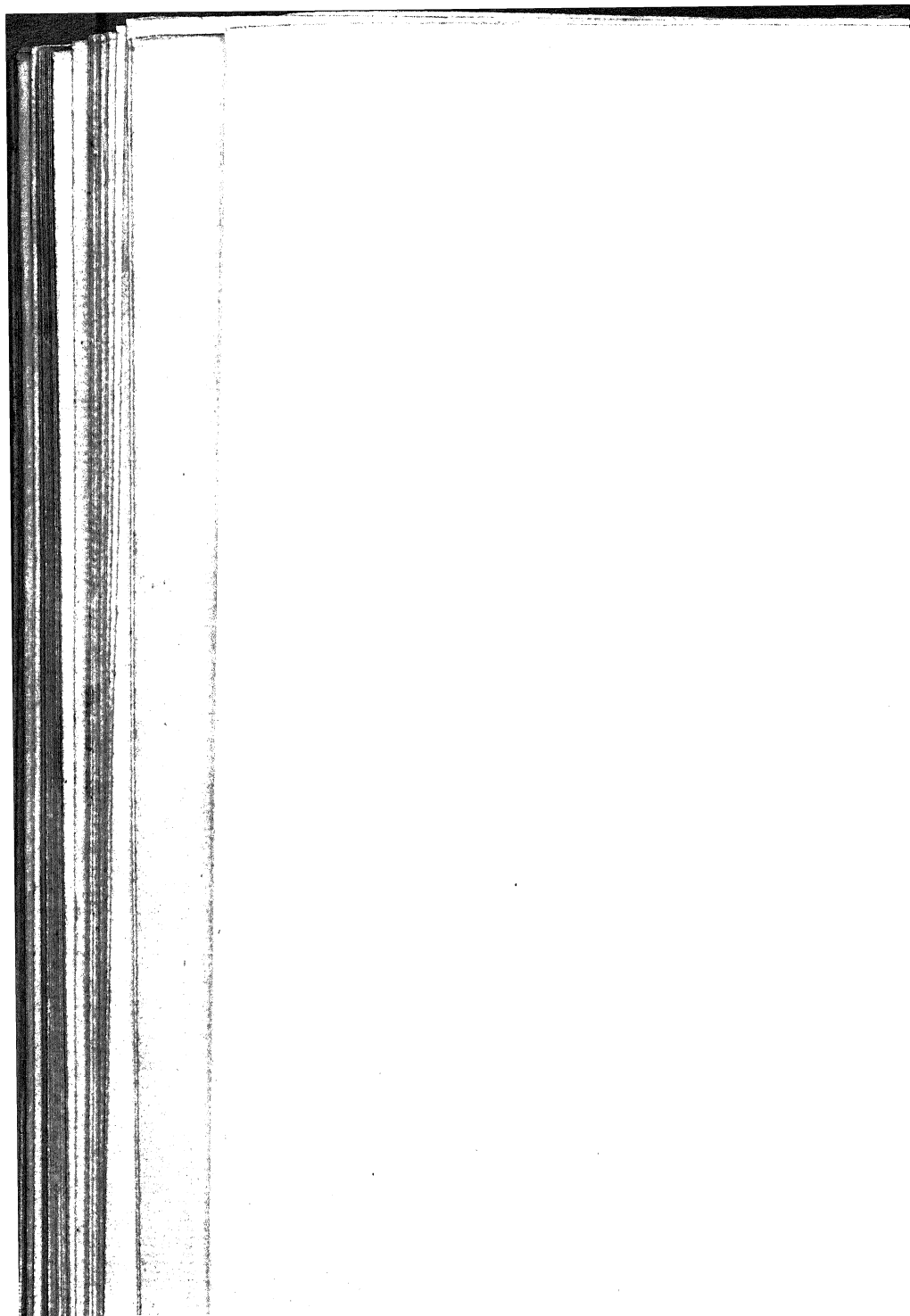
Fig. 93.

Part I. Let ABD , EFG be opposite hyperbolas, and let their asymptotes IP , RN cut one another in c ; c is the center of the hyperbola, or opposite hyperbolas.

In the curve of the hyperbola ABD take any two points A , D , and draw the straight line AD . Then AD will

226.





BOOK
III.

will meet both the asymptotes; for if it were parallel to either of the two, it would meet the curve in one point only, by Cor. 1. to the thirteenth Definition, and Prop. XIV. Book I. Let AD meet the asymptote RN in N , and the other in P ; and let LM , parallel to AD , touch the hyperbola ABD in B , and meet RN in L , and IP in M . Let NV , LX , MY , PW be straight lines parallel to the base of the cone in which the hyperbola was formed, and let them be supposed to have touched the conical superficies in the points V , X , Y , W ; and then, by Cor. 3. to the thirteenth Definition, NV , LX , MY , PW are equal. Also, by Cor. 1. Prop. XI. and Cor. 1. Prop. X. Book I. $BM^2 : MY^2 :: BL^2 : LX^2$; and $AP \times PD : PW^2 :: DN \times NA : NV^2$. Consequently (14. v.) BM^2 is equal to BL^2 , and $AP \times PD$ is equal to $DN \times NA$; and therefore BM is equal to BL , and, by Lemma VI. PD is equal to NA . Draw CB , and let it meet AD in O ; and then (4. vi.) $CB : CO :: BL : ON :: BM : OP$, and therefore (14. v.) ON is equal to OP . Consequently CB bisects AD , and for the same reasons it will bisect any straight line parallel to AD in the hyperbola; and therefore, by Cor. 1. Prop. III. Book II. CB is a diameter. Next, let the point H be in the curve of one hyperbola, and F in the other. Draw FH , and let it meet RN in Q , and IP in M . Let MY , AZ be two straight lines parallel to the base of the cone in which the sections were formed, and let them be supposed to have touched the conical superficies in Y and Z . Then, by Cor. 1. Prop. X. Book I. $HM \times MF : MY^2 :: HQ \times QF : AZ^2$, and therefore, by Cor. 3. to the thirteenth Definition (and 14. v.) $HM \times MF$ is equal to $HQ \times QF$, and, by Lemma VI. MH is equal to QF . Consequently if CS be drawn bisecting QM in S , it will bisect FH ; and in the same manner, as above, it may be proved,

BOOK
III.

proved, that cs will bisect any other straight line gk parallel to fh , and terminated by the opposite curves. By Cor. 1. Prop. III. Book II. cs is therefore a diameter, and consequently c is the center of the hyperbola.

Fig. 92.

Part II. Every thing remaining as in the thirteenth Definition, any straight line lv passing through the center c , and falling within the interior angle ick of the asymptotes, is a transverse diameter of the hyperbola.

For draw ac , and through ac , lcv let a plane pass, and let its line of common section with the vertical plane be az . Then (16. xi.) az , cv are parallel, as are also an , ck ; and therefore (10. xi.) the angles zan , $vcck$ are equal. In the same manner it may be demonstrated that the angles maz , icv are equal, and that the angle man is equal to the angle ick . The straight line az therefore, passing through a the vertex of the cone, falls within the opposite superficies, and consequently, as in Part II. Prop. IX. Book I. it may be proved, that lcv meets the opposite conical superficies. The straight line lcv therefore, passing through c the center, meets the curves of the opposite hyperbolas, and consequently is a transverse diameter.

Fig. 93.

Part III. Any straight line cs , passing through c and falling within rcp , the exterior angle of the asymptotes, is a second diameter of the hyperbola.

For if through o , any point within the hyperbola abd , a straight line as ad be drawn parallel to cs , it will meet the asymptotes, by Lemma III. and therefore, as it must cut the curve in two points, by Cor. 1. Prop. II. Book II. it is a second diameter.

Fig. 93.

Cor. 1. If a straight line as lm touch an hyperbola and meet the asymptotes, its segments between the point of contact and the asymptotes will be equal.

And

And if a straight line cutting an hyperbola, or opposite hyperbolas, meet the asymptotes, its segments between the curve or curves and the asymptotes will be equal. This is evident from the above; for it was proved that NA is equal to PD , and HM to FQ .

BOOK
III.

Cor. 2. If through any point, as P , in either asymptote, a straight line, as KG , be drawn, meeting the opposite hyperbolas in K and G ; and a straight line, as PA , be drawn, meeting the curve of the hyperbola ABD in D and A ; the rectangle under KP , PG will be equal to the square of the semidiameter parallel to KG , and the rectangle under AP , PD will be equal to the square of the semidiameter parallel to AP . For let EB be the diameter parallel to KG , and, the rest remaining as above, let the straight line CU be parallel to the base of the cone in which the section was formed, and let it be supposed to have touched the conical superficies in U . Then, by *Cor. 3.* to the thirteenth Definition, PW , CU are equal, and, by *Prop. XII. Book I.* $KP \times PG : PW^2 :: BC \times CE$ or $CB^2 : CU^2$; and therefore, as PW^2 , CU^2 are equal, $KP \times PG$ is equal to CB^2 . Again, by *Prop. V. Book II.* $KP \times PG$ is to $AP \times PD$ as CB^2 to the square of the semidiameter parallel to AP ; and therefore (14. v.) the *Cor.* is evident.

Cor. 3. A straight line as LM , touching the hyperbola ABD in B , and meeting the asymptotes in L , M , is parallel and equal to the second diameter conjugate to the transverse diameter EB , passing through the point of contact. For it may be proved, as in the last *Cor.* that the square of BM is equal to the square of the semidiameter parallel to it. Consequently, by *Cor. 1.* preceding, and by *Cor. 3. Prop. III. Book II.* this *Cor.* is evident.

BOOK
III

PROP. XVI.

According as a transverse diameter of an hyperbola is greater, equal to, or less than its conjugate diameter, the interior angle of the asymptotes is an acute, a right, or an obtuse angle; and any other transverse diameter is greater, equal to, or less than its conjugate diameter.

Fig. 94.

Let GB be an hyperbola, of which c is the center, and $κC$, cI the asymptotes, and let AB be any transverse diameter, and DE the diameter conjugate to it; according as AB is greater, equal to, or less than DE , the interior angle $κCI$ of the asymptotes is an acute, a right, or an obtuse angle; and any other transverse diameter FG is greater, equal to, or less than HL the diameter conjugate to it.

For let NI , touching the hyperbola in B , the vertex of AB , meet the asymptotes in N , I ; and let KM , touching the hyperbola in G , the vertex of FG , meet the asymptotes in K , M . Then, by Cor. 1. and Cor. 2. Prop. XV. NI is bisected in B , and KM in G ; and NI is equal to DE , and KM equal to HL , and therefore, according as AB is greater, equal to, or less than DE , CB is greater, equal to, or less than BI . But if with B as a center and BI as a distance a circle be described, its circumference will pass through N , and according as CB is greater, equal to, or less than BI , it will pass between c and B , through c , or on the opposite side of c from B . Consequently (by 31. iii. and 21. i.) according as CB is greater, equal to, or less than BI , the angle $κCI$ is an acute, a right, or an obtuse angle. Again, if with G as a center and GM as a distance a circle be described, its circumference will pass through K , and (by 31. iii. and 21. i.) according as the angle $κCI$ is an acute, a right, or an obtuse angle, the circumference of the circle will pass between c and G , through

through c , or on the opposite side of c from g . Con-
sequently, according as $\angle c$ is an acute, a right, or an
obtuse angle, cg is greater, equal to, or less than gm .
The Proposition therefore is evident.

BOOK
III.

If two conjugate diameters of an hyperbola be equal,
or if the angle contained by the asymptotes be a right
one, it is called an *Equilateral Hyperbola*.

PROP. XVII.

*The rectangle under two straight lines drawn from a point
in the curve of an hyperbola to the asymptotes is equal to
the rectangle under two straight lines, parallel to them,
drawn from any other point in the curve of the same or
opposite hyperbola to the asymptotes.*

The rectangle under the two straight lines AE , AD ,
drawn from the point A in the curve of the hyperbola
 AV to the asymptotes CH , CK , is equal to the rect-
angle under the straight lines BF , BG , parallel to AE ,
 AD , drawn from the point B in the curve of the same
or opposite hyperbola to the asymptotes. Fig. 95.

For draw AB , and let it meet the asymptotes in H
and K . Then, as AE , BF are parallel, (4. vi.) $AE :$
 $BF :: HA : HB$; and as BG , AD are parallel, $BG :$
 $AD :: KB : KA$. But, by Cor. 1. Prop. XV. HA is
equal to KB , and therefore HB is equal to KA . Con-
sequently (11. v.) $AE : BF :: BG : AD^*$, and (16. vi.)
 $AE \times AD$ is equal to $BF \times BG$.

Cor. 1. If AE , BF be parallel to the asymptote CK ,
and AD , BG be parallel to the asymptote CH ; the pa-

* This is the property referred to by writers on Natural Philosophy,
when they prove, that the curve formed by the upper surface of a li-
quid raised by the force of attraction between two plates, meeting at
one end, and kept at a small distance from one another at the other, is
an hyperbola.

BOOK
III.

parallelograms ED, FG (14. vi.) are equal. For the angle at c being common to the two parallelograms, they are equiangular, and, by the above, the sides round the equal angles are reciprocally proportional.

Cor. 2. If from any two points as A, B in the curve of an hyperbola $A \vee B$ two straight lines AE, BF , parallel to one of the asymptotes as CK , be drawn to the other asymptote CH ; then $CF : CE :: EA : FB$. And the semitransverse diameters CA, CB being drawn, the triangles CFB, CEA are equal.

Cor. 3. If CH, CK be the asymptotes of an hyperbola $A \vee B$, and if from any two points F, E in CH , straight lines FB, EA be drawn parallel to CK , and if FB be drawn to the curve and EA towards it, and if EC be to CF as FB to EA , the point A must also be in the curve.

DEFINITIONS.

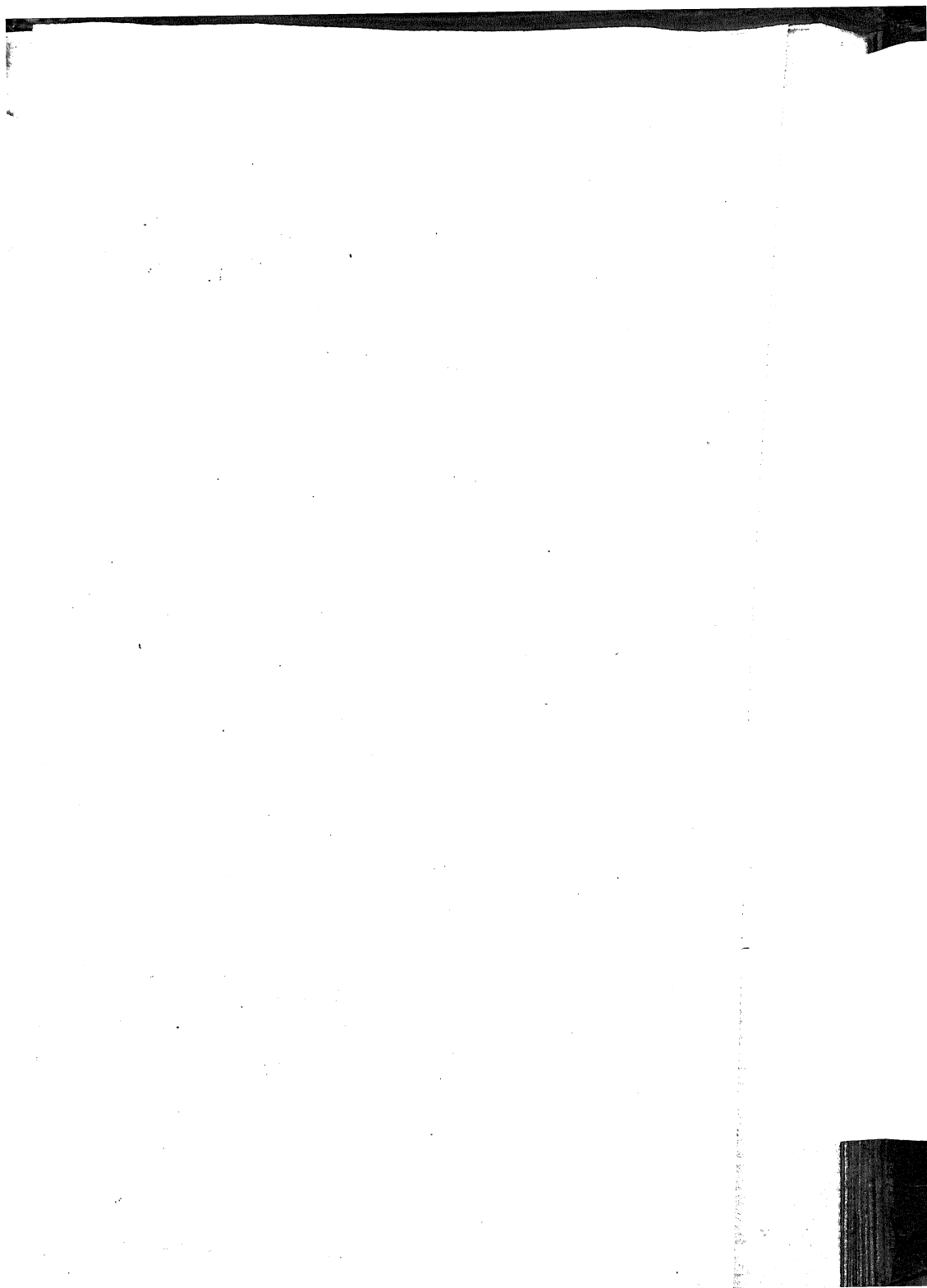
XV.

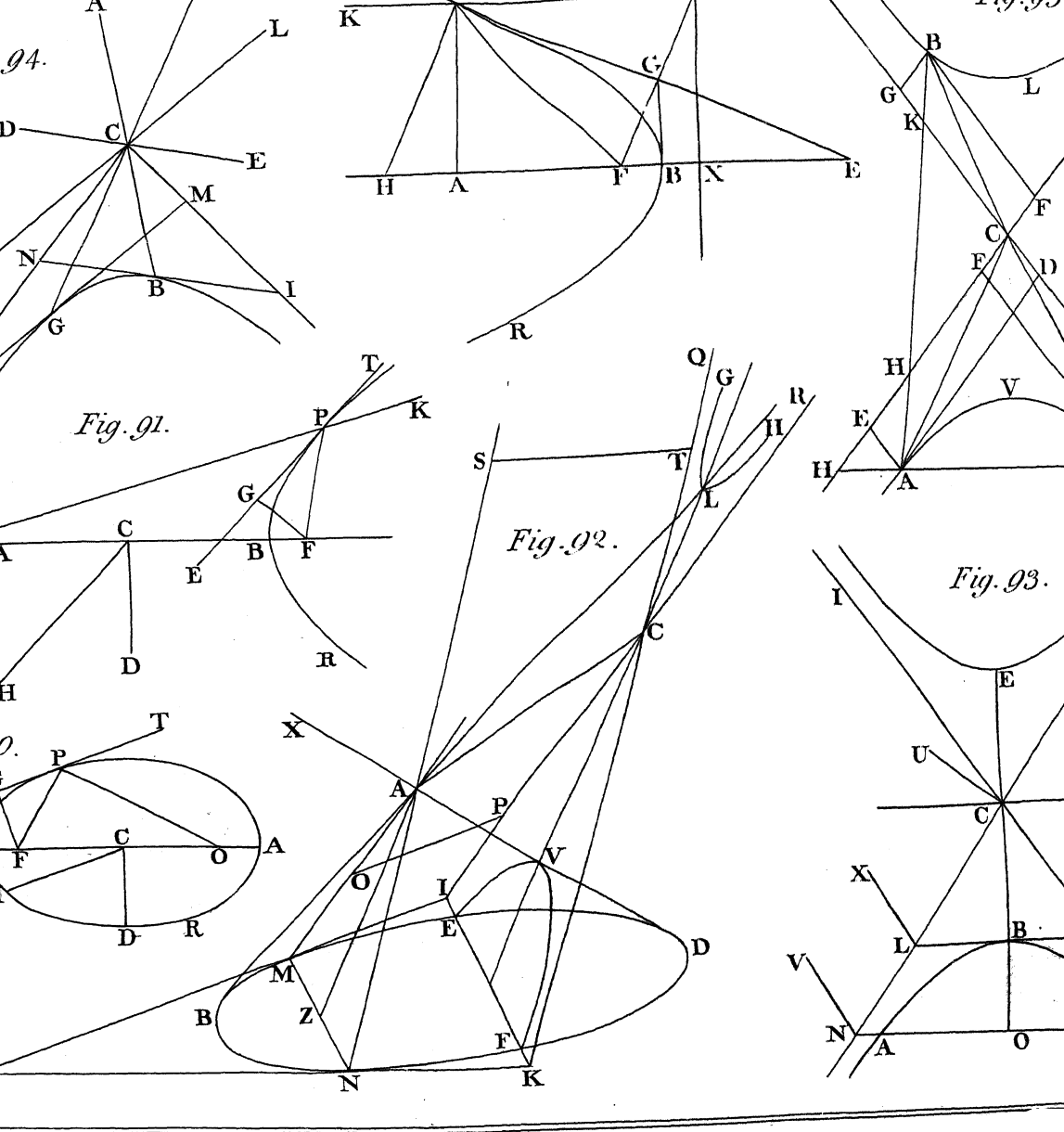
Fig. 96. If from c the center of the hyperbola $M N \alpha$ any two semitransverse diameters $cN, c\alpha$ be drawn to the curve, the figure $cN\alpha$ bounded by the femidiameters and the curve $N\alpha$ is called a *Hyperbolic Sector*.

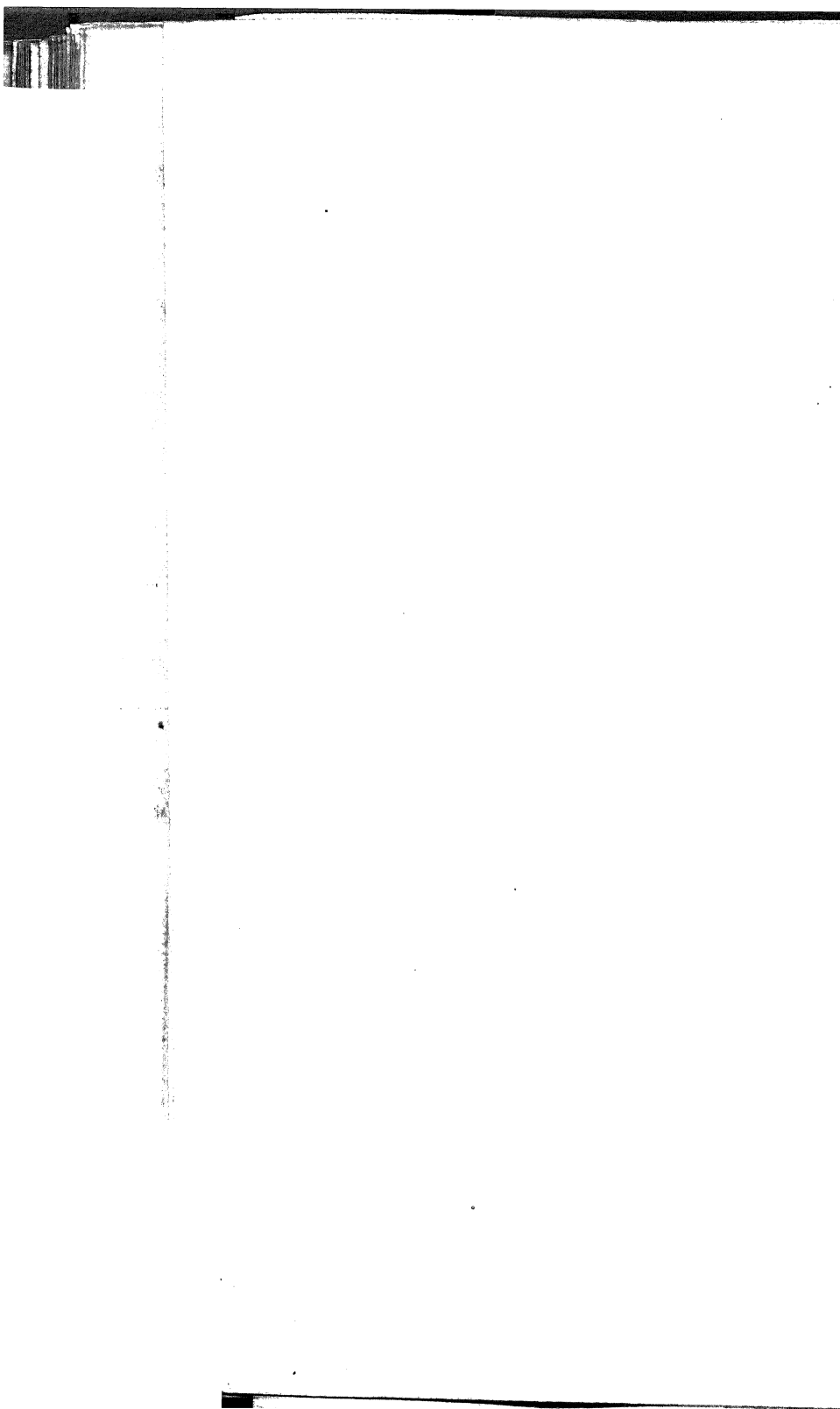
XVI.

If cR, cG be the asymptotes of the hyperbola $M N \alpha$, and from any two points N, α in the curve, straight lines $NB, \alpha A$ parallel to the asymptote cR be drawn to the other asymptote cG , the figure $ABN\alpha$, bounded by the straight lines $NB, BA, \alpha A$ and the curve $N\alpha$, is called a *Hyperbolic Trapezium*.

Cor. A hyperbolic sector $cN\alpha$ and trapezium $ABN\alpha$ upon the same curve are equal. For let cN cut αA in P , and then, as by *Cor. 2.* *Prop. XVII.* the triangles $CA\alpha, CBN$ are equal, the rectilineal trapezium $ABNP$ is equal to the triangle $CP\alpha$. To these equals add the







the figure $P N a$, and then the hyperbolic sector $c N a$ is equal to the hyperbolic trapezium $A B N a$.

BOOK
III.

XVII.

Any segment as $c A$, intercepted between c the center and A a point in either asymptote, is called an *Asymptotic Segment*, and the point A is called its extremity.

XVIII.

Any straight line in the plane of an hyperbola, or opposite hyperbolas, parallel to either of the asymptotes is called an *Asymptotic Secant*.

PROP. XVIII.

If from the points in which a straight line cuts, and the point in which a straight line parallel to it touches, an hyperbola, straight lines parallel to one of the asymptotes be drawn to the other, they will cut off from the center proportional asymptotic segments: and, on the contrary, if from the extremities of three proportional asymptotic segments straight lines parallel to the other asymptote be drawn to the curve of the hyperbola, the straight line joining the extreme points in the curve will be parallel to the tangent passing through the middle point in the curve.

Let the straight line $M a$ cut the hyperbola $M N a$ in the points M , a , and let the straight line $G R$, parallel to it, touch the hyperbola in N , and from a , N , M let the straight lines $a A$, $N B$, $M D$, parallel to the asymptote $c R$, be drawn to the asymptote $c G$; the asymptotic segments $c A$, $c B$, $c D$ are proportionals. On the contrary, if $c A$, $c B$, $c D$ be proportional asymptotic segments, and straight lines $A a$, $B N$, $D M$, parallel to the asymptote $c R$, be drawn to meet the curve of the hyperbola in a , N , M , the straight line

BOOK III. MA , joining the extreme points, is parallel to GR touching the hyperbola in the middle point N .

Part I. Let the secant MA meet the asymptotes CG , CR in the points H , K , and let the tangent GR , parallel to the secant, meet them in G , R . Then, as DM , AG are parallel, (10. vi.) $HM : HD :: KA : CA$, and therefore, by Cor. 1. Prop. XV. (and 14. v.) CA is equal to DH . Again, as by Cor. 1. Prop. XV. GN , NR are equal, and as BN , CR are parallel, (2. vi. and 14. v.) CB , BG are equal. Consequently $CA : CB :: DH : BG$; and (4. vi.) $DH : BG :: DM : BN$, and therefore (11. v.) $CA : CB :: DM : BN$. But, by Cor. 2. Prop. XVII. $DM : BN :: CB : CD$; and therefore (11. v.) $CA : CB :: CB : CD$.

Part II. Upon the second hypothesis let MA meet the asymptotes in H , K , and let the tangent passing through N meet them in G , R . Then, as above, it may be demonstrated that CA is equal to DH , and CB equal to BG ; and therefore $CA : CB :: DH : BG$. But, by hypothesis, $CA : CB :: CB : CD$; and, by Cor. 2. Prop. XVII. $CB : CD :: DM : BN$. Consequently, (11. v.) $DH : BG :: DM : BN$, and by alteration $DH : DM :: BG : BN$; and therefore as DM , BN are parallel, the angles (6. vi.) DHM , BGN are equal, and (29. i.) HK , GR are parallel.

Cor. If two parallel straight lines MA , LF cut an hyperbola $LMQF$, and from the points F , A , M , L straight lines FE , QA , MD , LO parallel to the asymptote CT be drawn to the other asymptote CS , the asymptotic segments CE , CA , CD , CO will be proportional. On the contrary, if the asymptotic segments CE , CA , CD , CO be proportional, and EF , AQ , DM , LO , parallel to the asymptote CT , be drawn to the curve, the straight line LF joining the extreme points will be parallel to MA joining the mean points. For,
first,

first, if GR , parallel to MA or LF , touch the hyperbola in N and NB , parallel to the asymptote CT , be drawn to the other asymptote CS , then by the above (and 17. vi.) $CA \times CD$ is equal to CB^2 , and also $CE \times CO$ is equal to CB^2 . Consequently $CE \times CO$ is equal to $CA \times CD$, and $CE : CA :: CD : CO$.

Upon the second hypothesis let CB be a mean proportional between CA , CD , and let BN , parallel to CT , be drawn to the curve, and let GR touch the hyperbola in N . Then, by the second Part of the above, GR , MA are parallel. Again, as by hypothesis $CE : CA :: CD : CO$, $CE \times CO$ is equal to $CA \times CD$. But CB being a mean proportional between CA , CD , CB^2 is equal to $CA \times CD$, and therefore $CE \times CO$ is equal to CB^2 , and CB is a mean proportional between CE , CO . Consequently as before LF is parallel to GR , and therefore (30. i.) MA , LF are parallel.

PROP. XIX.

If from the extremities of four proportional asymptotic segments asymptotic secants be drawn to the curve of the hyperbola, the hyperbolic trapezium between the first and second secant will be equal to the hyperbolic trapezium between the third and fourth. And if from the extremities of a series of asymptotic segments, in geometrical progression, asymptotic secants be drawn to the curve of the hyperbola, the hyperbolic trapezia between the first and second secant, the first and third, the first and fourth, and so on, will be in arithmetical progression.

Part I. Let MNA be an hyperbola, of which c is the center and CT , CS the asymptotes, and in CS let CE be to CA as CD to CO , and let EF , AQ , DM , OL be asymptotic secants drawn to the curve, the hyper-

Fig. 96.

BOOK bolic trapezium $E A Q F$ is equal to the hyperbolic tra-
III. pezium $D O L M$.

For $M Q$, $L F$ being drawn, they will be parallel, by Cor. Prop. XVIII. Draw $c L$, $c M$, $c Q$, $c F$. Let $c V$ be the diameter to which the parallels $M Q$, $L F$ are double ordinates, and let it meet $M Q$ in T , and $L F$ in V . Then (38. i.) the triangle $c L V$ is equal to the triangle $c F V$, and the triangle $c M T$ is equal to the triangle $c Q T$; and as the diameter $c V$ bisects all straight lines parallel to $M Q$ and terminated by the curve, the space $T M L V$ is equal to the space $T Q F V$. Consequently (axiom 3. i.) the hyperbolic sector $c F Q$ is equal to the hyperbolic sector $c M L$; and therefore, by Cor. to the sixteenth Definition, the hyperbolic trapezia $E A Q F$, $D O L M$ are equal.

Part II. The rest remaining as above, let $c A$, $c B$, $c D$, $c X$, &c. be a series of asymptotic segments in geometrical progression, and let the asymptotic secants $A Q$, $B N$, $D M$, $X Y$, &c. be drawn to the curve; the hyperbolic trapezia $A B N Q$, $A D M Q$, $A X Y Q$, &c. are in arithmetical progression.

For let $G R$ touch the hyperbola in N , and then, by Prop. XVIII. it is parallel to $M Q$. Let the diameter $c T$ pass through N , and then, by Prop. II. it bisects $M Q$ in T ; and (38. i.) the triangles $c T Q$, $c T M$ are equal. And as $c T$ bisects every straight line parallel to $M Q$, and terminated by the curve, the space $N M T$ is equal to the space $N Q T$. Consequently (axiom 3. i.) the hyperbolic sectors $c A N$, $c N M$ are equal; and therefore, by Cor. to the sixteenth Definition, the hyperbolic trapezia $A B N Q$, $B D M N$ are equal. As, by hypothesis, $c B$ is to $c D$ as $c D$ to $c X$, it may be proved, in the same way, that the hyperbolic trapezia $B D M N$, $D X Y M$ are equal; and the same mode of proof may be extended to any number of terms. Con-
 sequently

frequently the hyperbolic trapezia $ABNQ$, $ADMQ$, $BOOK$
 $AXYQ$, &c. are in arithmetical progression. III.

SCHOLIUM.

As the hyperbola and its asymptotes may be indefinitely extended, it is evident that a series of asymptotic segments in geometrical progression, and a corresponding series of hyperbolic trapezia in arithmetical progression, may be continued to any assigned number of terms. From the nature of logarithms, therefore, the series of asymptotic segments CA , CB , CD , &c. as above, is analogous to a series of natural numbers in geometrical progression, and the series of hyperbolic trapezia $ABNQ$, $ADMQ$, &c. as above, is analogous to the logarithms of these natural numbers. To enter into an explanation of these analogies would be incompatible with the design of this work. The reader may find full information on the subject of logarithms in the volumes entitled, "*Scriptores Logarithmici*," published by Francis Maseres, Esq. F. R. S. Curfitor Baron of the Court of Exchequer. To this Gentleman the mathematical world are highly indebted for his persevering exertions and liberality in the cause of science. He has employed his great abilities in endeavours to render some of the most important subjects perspicuous, and he has expended large sums in the publication of scarce mathematical tracts, and made presents of many copies of them, with the highly laudable motive of promoting learning and diffeminating knowledge.

Within these twenty years, last past, much has been done in this country to facilitate the application of logarithms, and to extend their utility. In 1785 Dr. Hutton of Woolwich published in 8vo. extensive Tables of them, to which he prefixed "A large and original History of the Discoveries and Writings relating
to

BOOK III. to those subjects." These Tables are judiciously arranged, and are very valuable for general use. The history prefixed to them is a masterly performance; it gave rise to the publication mentioned above, entitled, "Scriptores Logarithmici."

In the year 1792 a quarto volume was published, under the patronage of the Board of Longitude, entitled, "Tables of Logarithms of all Numbers, from 1 to 101000; and of the Sines and Tangents to every second of the Quadrant. By Michael Taylor, Author of the Sexagesimal Table." As Mr. Taylor died before the Tables were entirely printed, the Rev. Dr. Maskelyne, Astronomer Royal, superintended their completion. He also wrote the Preface and Precepts for the use of the Tables; and these he executed with that care, learning, and ability, for which he is so justly celebrated in every part of the world where either the theory, or practical utility, of Astronomy is understood. As Taylor's Tables are accurate, and more extensive than any other extant, the volume is an excellent resource to those who aim at a superior degree of precision in their calculations.

In a small quarto volume of mathematical Essays, published in 1788 by the Rev. John Hellins, (now Vicar of Potters' Pury, Northamptonshire, and F.R.S.) there are two Essays on Logarithms, which display an intimate knowledge of the subject. A scientific reader will find much gratification in the perusal of these Essays.

DEFINITION.

XIX.

Fig. 97. If AB be a transverse diameter of the opposite hyperbolas A, B , and DE the second diameter conjugate to it, and if DE be a transverse diameter of the opposite

posite hyperbolas D, E , and $A B$ the second diameter conjugate to it; the hyperbolas D, E are called the *Conjugate Hyperbolas* to one or both of the opposite hyperbolas A, B , and, on the contrary, the hyperbolas A, B are called the *Conjugate Hyperbolas* to one or both of the opposite hyperbolas D, E . When all the four hyperbolas A, D, B, E are mentioned together, they are called *Conjugate Hyperbolas*.

Cor. If the diameters $A B, D E$ cut one another in C , it is evident that C is the common center of the conjugate hyperbolas.

PROP. XX.

One of the asymptotes of an hyperbola is parallel to, and the other bisects, a straight line joining the vertices of any two conjugate diameters: and the vertices of second diameters of an hyperbola are in the curves of the hyperbolas conjugate to it.

Part I. Let $A B, D E$ be any two conjugate diameters of the hyperbola $A H$, and let $C F, C G$ be its asymptotes, C being the center; one of the asymptotes, as $C F$, is parallel to $A E$ the straight line joining the vertices A, E , and the other asymptote $C G$ bisects $A E$. Fig. 97.

For let $F G$ touch the hyperbola $H A$ in A and meet the asymptotes in F and G . Then, by Cor. 3. and I. Prop. XV. $F G$ is equal and parallel to $D E$, and $C E$ is equal to $F A$ and also to $A G$. Consequently (33. i.) the asymptote $C F$ is parallel to $A E$. Also the angle (29. i.) $C E L$ is equal to the angle $L A G$, and the angle $E C L$ to the angle $A G L$, and therefore (26. i.) $A L$ is equal to $E L$, and $A E$ is bisected by the asymptote $C G$.

Part II. Let $K H$ be any transverse diameter of the opposite hyperbolas $A H, B K$, and let $M N$ be the second

BOOK III.
 cond diameter conjugate to it; the vertices M, N of this second diameter are in the curves of the hyperbolas conjugate to the hyperbolas AH, BK .

For let D, E be the hyperbolas conjugate to AH, BK , and let AB, DE be the conjugate diameters common to the conjugate hyperbolas, as in the nineteenth Definition, and c the center. Let FC, GC be the asymptotes of the opposite hyperbolas A, B . Draw BE, KN , and let them meet the asymptote FC in P and Q ; and then, by Part I. BE, KN are parallel to the asymptote GC , and they are bisected by the asymptote FC in P and Q . Consequently, by Cor. 2. Prop. XVII. $CP : CQ :: KQ : BP$; and therefore, on account of the equals, $CP : CQ :: QN : PE$, and as E is in the curve of the hyperbola E, N must be in the curve of the same hyperbola E , by Cor. 3. Prop. XVII.

Cor. A straight line parallel to one of the asymptotes, and terminated by the curves of the conjugate hyperbolas, is bisected by the other asymptote.

PROP. XXI.

A quadrilateral figure, whose sides pass through the vertices of any two conjugate diameters of an ellipse, or conjugate hyperbolas, and touch the ellipse or hyperbolas, is a parallelogram, and is equal to the rectangle under the axes of the ellipse or hyperbolas.

Fig. 98.
 99. Let $MSLR$ be a quadrilateral figure, whose sides MS, SL, LR, RM pass through F, G, H, K , the vertices of the conjugate diameters FH, GK of the ellipse $FGHK$, or the conjugate hyperbolas F, G, H, K , and in these points touch the curve or curves; the figure $MSLR$ is a parallelogram, and is equal to the rectangle under the axes AB, DE of the ellipse or hyperbolas.

For,

For, by Cor. 2. Prop. IV. Book II. the tangents MS , RL are parallel to the diameter KG , and the tangents MR , LS are parallel to the diameter FH . The quadrilateral figure $MSLR$ is therefore a parallelogram.

Again, let c be the center, and let CI be perpendicular to the tangent SL . Then as FH is bisected in c , the parallelogram GH is a fourth part of the parallelogram LM ; and, by Cor. 1. Prop. XIX. Book II. $CI : CB :: CD : CH$, and $CI \times CH$ is equal to $CB \times CD$. But (35. i.) $CI \times CH$ is equal to the parallelogram GH , and $CB \times CD$ is a fourth part of the rectangle under the axes AB , DE . Consequently the parallelogram LM is equal to the rectangle under the axes AB , DE .

Cor. All parallelograms contained under tangents, passing through vertices of conjugate diameters of an ellipse, or conjugate hyperbolas, are equal to one another, and each of them is equal to the rectangle under the axes.

The twelfth Lemma of Sir Isaac Newton's Principia, Lib. I. and the tenth Proposition, depending upon it, are evident from the above.

PROP. XXII.

If through a point in the curve of an hyperbola two straight lines be drawn parallel to the asymptotes and meeting a diameter, the semidiameter will be a mean proportional between the segments of the diameter between the center and the points of concurrence.

Through the point I in the curve of the hyperbola Fig. 100.
 IB let the straight lines IT , IA be drawn parallel to the asymptotes CE , CG , and first let them meet the transverse diameter DB in the points T , A , and let c be the

BOOK the center ; the femidiameter CB is a mean proportional between CT , CA .

III.

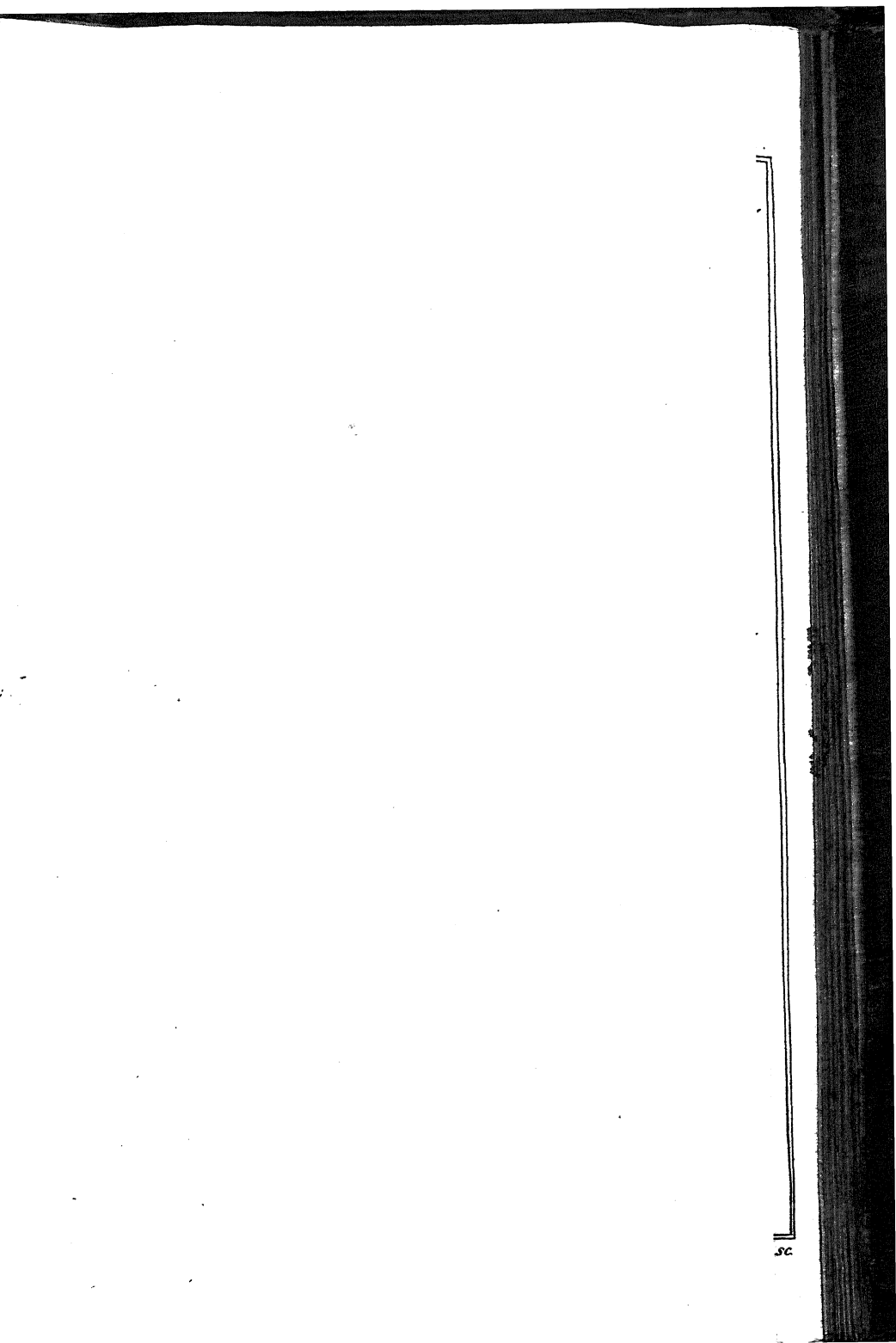
For let IA meet the asymptote CE in F , and let BH , TK be parallel to IF , or to the asymptote CG , and let them meet CE in H and K . Then (34. i.) KT , FI are equal ; and (4. vi.) $CK : CH :: KT$ or its equal $FI : HB$. But, by Cor. 2. Prop. XVII. $FI : HB :: CH : CF$, and therefore (11. v.) $CK : CH :: CH : CF$. Consequently, on account of the parallels KT , HB , FA , $CT : CB :: CB : CA$.

Secondly, the rest remaining as above, let the straight lines IT , IA meet the second diameter LM in the points N , O , and let IA meet the curve of the hyperbola PL , conjugate to BI , in P ; and let PQ be parallel to CE or IN , and meet the diameter LM in Q . Then, by the above, $CQ : CL :: CL : CO$. But, by Cor. Prop. XX. IF , FP are equal, and therefore, as PQ , IN are parallel, CQ is equal to CN . Consequently $CN : CL :: CL : CO$.

Fig. 101.
102.

Cor. 1. If two straight lines TN , TP , touching an hyperbola, or opposite hyperbolas, in N and P , meet one another in T , and if a straight line TI parallel to one of the asymptotes meet the curve in I , a straight line IA parallel to the other asymptote and meeting NP , the straight line joining the points of contact, in A will bisect NP in A . This is evident from the above, and Prop. VII. Book II.

Cor. 2. The rest remaining as in the preceding Cor. if TI produced meet NP in E , TE will be bisected in I . For let TL parallel to IA , or an asymptote, meet the curve in L , and then, by Cor. 1. LA parallel to TI will meet NP in A , the point in which NP is bisected ; and $TIAL$ is a parallelogram. Draw IL , and let it meet the diameter CBA in G . Then (34. i.) IA , TL are equal, and (29. i.) the triangles $AI G$, $TL G$ are equi-



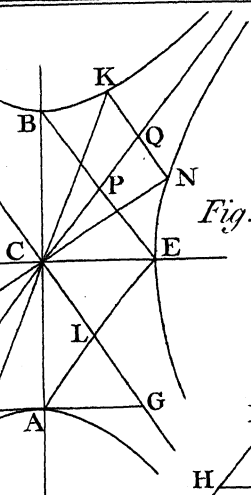


Fig. 97.

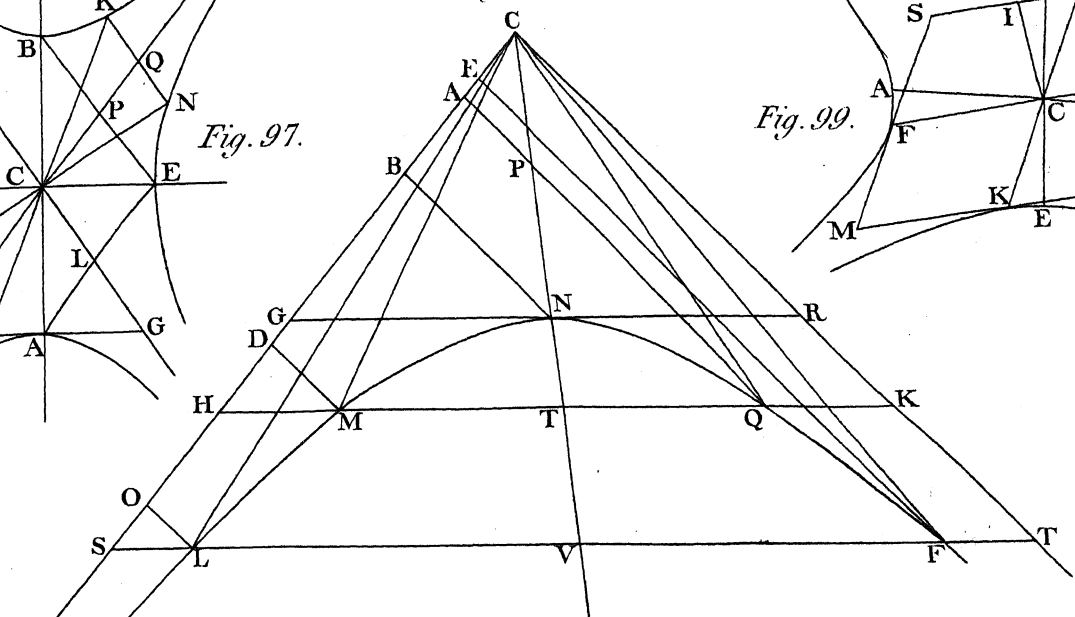


Fig. 96.

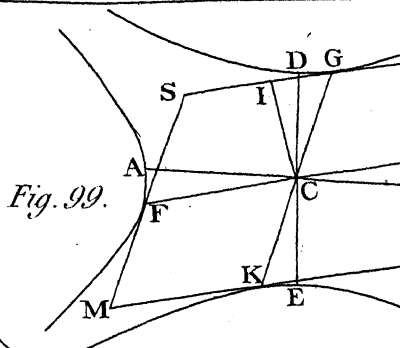


Fig. 99.

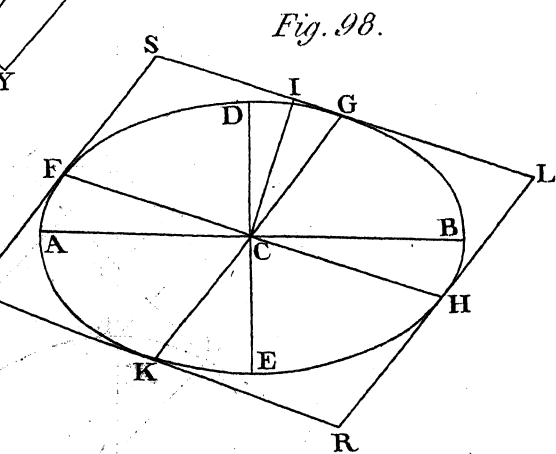


Fig. 98.

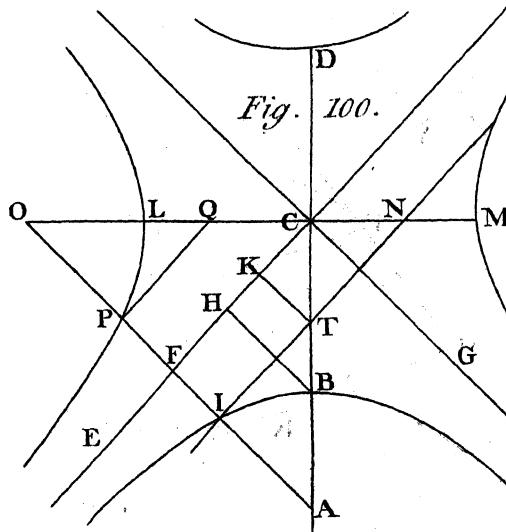
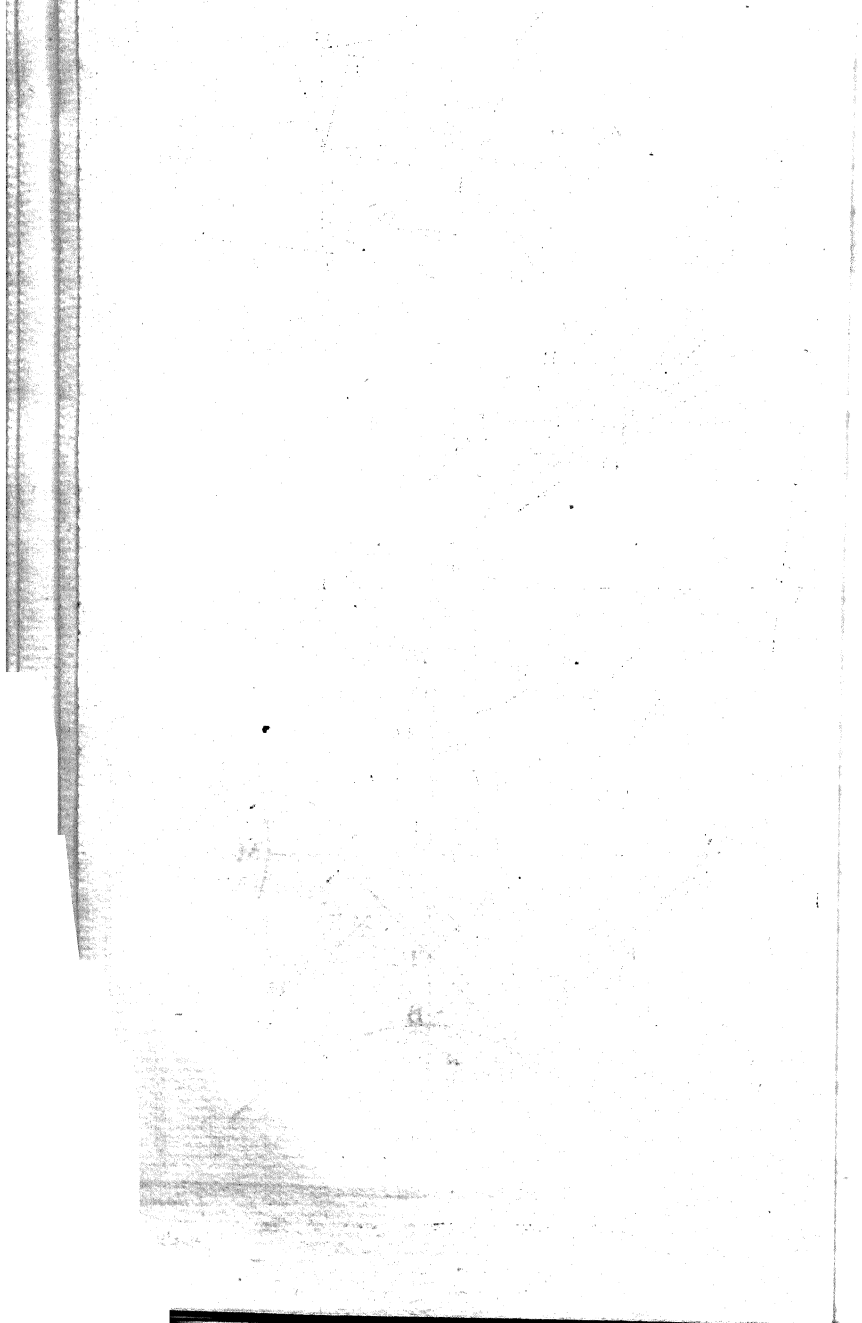


Fig. 100.



equiangular; and therefore (26. i.) IG, GL are equal, and TG is equal to GA . The straight lines IL, NP are therefore ordinates to the diameter CBA , and consequently, by Prop. II. they are parallel; and (2. vi.) $TG : GA :: TI : IE$. The straight line TE is therefore bisected in I .

BOOK
III.

PROP. XXIII.

If a diameter of a parabola, or a straight line parallel to an asymptote of an hyperbola, meet two tangents and the straight lines joining the points of contact, the square of its segment between the curve and the straight line joining the points of contact will be equal to the rectangle under the segments between the curve and tangents.

If the diameter of the parabola, or the straight line parallel to an asymptote of an hyperbola, pass through the point in which the tangents meet one another, the Proposition is evident from Prop. V. and Cor. 2. Prop. XXII: but if not, let the two straight lines MN, MO touch the parabola, hyperbola, or opposite hyperbolas, in the points N, O , and let TX a diameter of the parabola, or a straight line parallel to an asymptote, meet the parabola or either hyperbola in E , the tangents in A, D , and the straight line joining the points of contact in B ; the square of EB is equal to the rectangle under AE, ED .

Fig. 103.
104.
105.

Case I. If the straight lines touching the parabola or hyperbola meet one another in M , from the point D in which TX meets one of the tangents draw DL parallel to the other tangent MO , and let it meet the curve in P, L , and the straight line NO in K . Then, by Prop. XVII. Book I. the square of DK is equal to the rectangle under LD, DP ; and (4. and 22. vi.)

Fig. 103.
104. AB^2

BOOK III. $AB^2 : DB^2 :: AO^2 : DK^2$ or $LD \times DP$. But as TX is parallel to a side of the cone in which the section was formed, by Prop. XVI. Book I. $AO^2 : LD \times DP :: AE : DE$; and therefore (II. v.) $AE : DE :: AB^2 : DB^2$. Consequently, by Lemma VIII. $AE : BE :: BE : DE$; and (17. vi.) $AE \times ED$ is equal to BE^2 .

Fig. 105. Case 2. If the straight lines touching the opposite hyperbolas meet one another in M , from A or D , suppose D , draw the straight line DK parallel to the tangent AO , and let it meet NO in K . Through the points A, D draw the straight lines GH, PL parallel to one another, and let them meet the opposite hyperbolas in the points G, H and P, L . Then, by Prop. V. Book II. $GA \times AH$ is to AO^2 as the square of the semidiameter parallel to GH to the square of the semidiameter parallel to AO . And, by Cor. I. Prop. XVII. Book I. and Prop. V. Book II. $PD \times DL$ is to DK^2 as the squares of the same semidiameters. Consequently (II. v. and alternation) $GA \times AH : PD \times DL :: AO^2 : DK^2$. But (4. and 22. vi.) $AO^2 : DK^2 :: AB^2 : DB^2$; and, by Prop. XVI. Book I. $GA \times AH : PD \times DL :: AE : DE$. Consequently $AE : DE :: AB^2 : DB^2$, and therefore, by Lemma VIII. $AE : BE :: BE : DE$, and $AE \times DE$ is equal to BE^2 .

Fig. 105. Case 3. If the straight lines AO, DN touching the opposite hyperbolas be parallel, then the triangles ABO, DBN will be equiangular, and the proportion will be $AO^2 : DN^2 :: AB^2 : DB^2$; and in this case, by Prop. XVI. Book I. $AE : DE :: AO^2 : DN^2$. Consequently $AE : DE :: AB^2 : DB^2$, and, by Lemma VIII. $AE : BE :: BE : DE$, and $AE \times DE$ is equal to BE^2 .

PROP. XXIV.

If from two given points in the curve of a parabola, or hyperbola, or the curves of opposite hyperbolas, two straight

Straight lines be inflected to any third point in the curve of the parabola, or in the curve of either of the opposite hyperbolas, and if they meet a diameter of the parabola, or a straight line parallel to an asymptote of the hyperbola; the segments of this last mentioned line, between the inflected lines and the point in which it meets the curve of the section, will be to one another in the same ratio, wherever the point may be in the curve to which the lines are inflected.

Let N, M be two given points in the curve of the parabola, hyperbola, or opposite hyperbolas, and let the straight lines NO, MO inflected from them to any point O in the curve of the parabola, or in the curve of either hyperbola, meet in B, C the straight line TX , a diameter of the parabola, or parallel to an asymptote of the hyperbola, and let TX meet the curve in E ; the segments EB, EC are to one another in the same ratio, wherever the point O may be taken in the curve.

Fig. 106.
107.
108.

For let tangents passing through M, N, O meet TX in F, D, A . Draw MN , and let it meet TX in G ; and by Prop. XXIII. $ED : EG :: EG : EF$. Again, by Prop. XXIII. EB^2 is equal to $AE \times ED$, and EC^2 is equal to $AE \times EF$. Consequently $EB^2 : EC^2 :: AE \times ED : AE \times EF :: (I. vi.) ED : EF$. But, by the above, (and Cor. 2. 20. vi.) $ED : EF :: ED^2 : EG^2$; and therefore (II. v.) $EB^2 : EC^2 :: ED^2 : EG^2$, and (22. vi.) $EB : EC :: ED : EG$. But as the points M, N are given, or fixed, and as the straight line TX is given in position, the segments ED, EG remain the same. Consequently, if the point O be moved round the curve of the parabola, hyperbola, or opposite hyperbola, the segments EB, EC of the straight line TX , between the inflected lines and the curve, will be to one another in the same ratio, wherever the point O may be in the curve.

BOOK
III.

SCHOLIUM.

It may be proper in this place to direct the attention of the reader to methods of ascertaining certain particulars in a conic section, supposing the curves of the sections to be given, or straight lines to be given, for the description of the curves. These methods, now to be described, might have been delivered as Corollaries to the Propositions on which they depend, or they might have been put into the form of Problems; but it appeared more advisable to reserve them for a series of articles in a Scholium. A cautious desire against interrupting the reader in the acquisition of new truths suggested this delay. The ease with which they are deduced from the preceding Propositions, and the importance of the articles themselves, induced the author to think that the following was the most proper manner of delivering them, and the most proper place to insert them.

Fig. 109.
110.

1. Let the ellipse $A D B E$, or the opposite hyperbolas $A, D B E$, be given to find the center.

In the ellipse and in either hyperbola draw the two parallel straight lines $D E, F G$, and draw $A B$ bisecting $D E$ in H and $F G$ in K . Let $A B$ meet the curve of the ellipse, or the curves of the opposite hyperbolas, in A and B . Bisect $A B$ in C , and C will be the center, by Cor. 1. Prop. III. Book II.

If only one hyperbola $D B E$ be given, two other straight lines must be drawn parallel to one another, but not parallel to $D E, F G$, and a straight line being drawn bisecting them will be a diameter. Its concurrence therefore with $A B$ will determine the center.

2. The curve of a conic section and a point in it being given, let be required to draw a diameter through the given point.

If the section be an ellipse, or hyperbola, find the
cen-

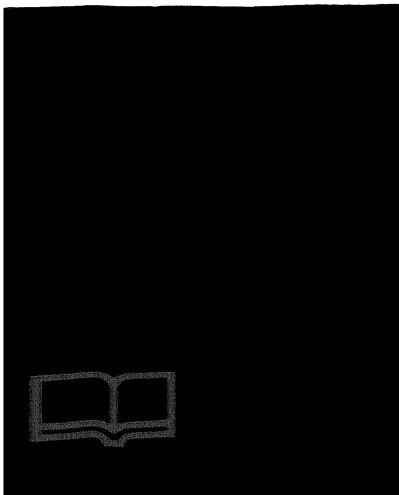


Fig. 102.

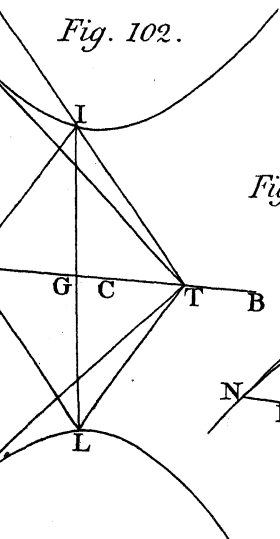


Fig. 101.

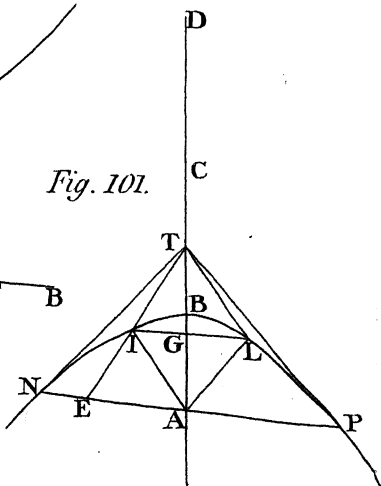


Fig. 106.

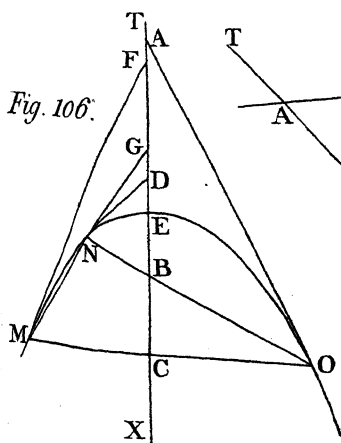


Fig. 107.

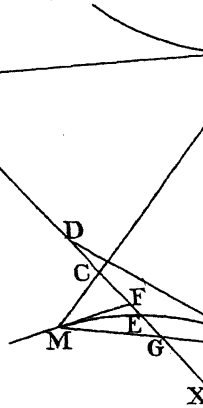


Fig. 103.

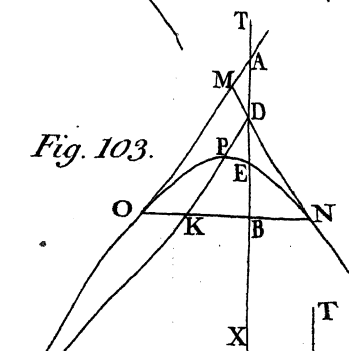


Fig. 105.

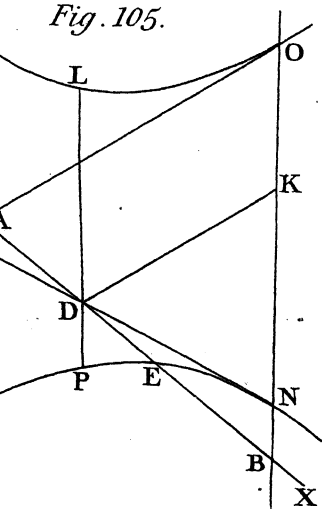


Fig. 104.

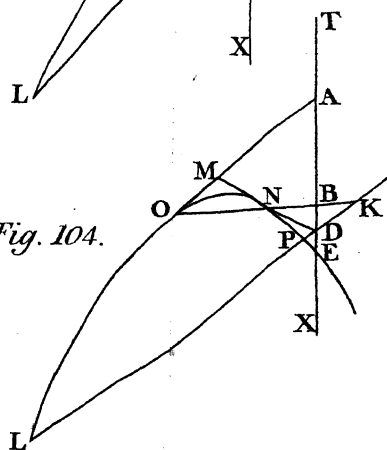
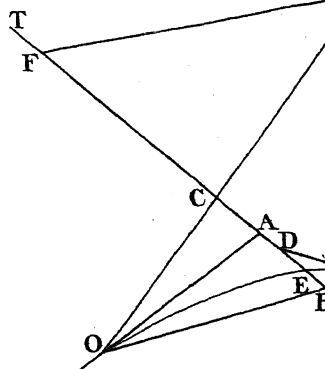
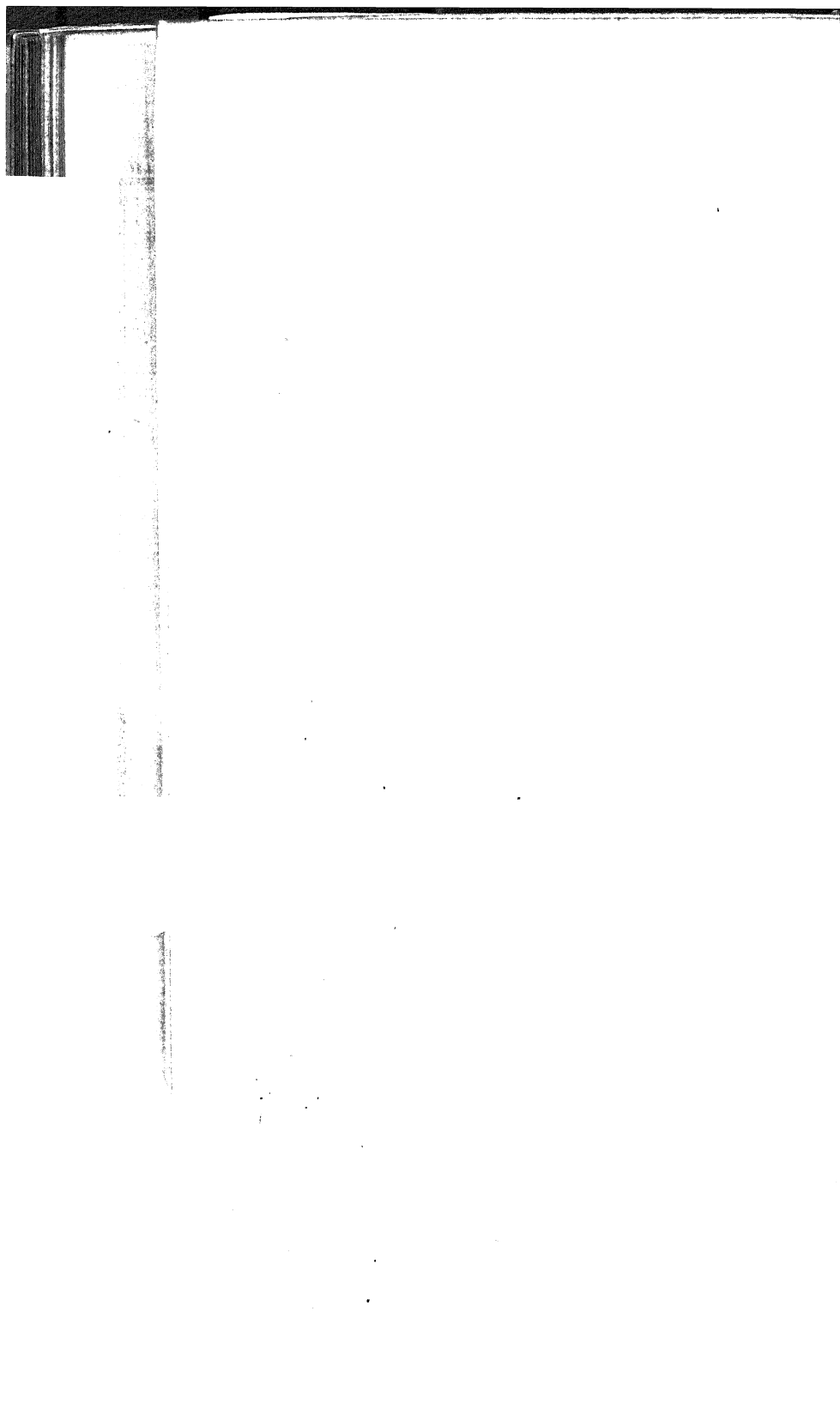


Fig. 108.





center by the preceding article, and through the center and the given point draw a diameter. If the section be a parabola, find a diameter, by Cor. 3. Prop. II. of this Book, and parallel to it draw a straight line through the given point; and, by Cor. 1. to the first Definition, this will be the diameter required. BOOK III.

3. The curve and a diameter AB of a conic section being given, and any point G in the curve besides a vertex of the given diameter, let it be required to draw a straight line from G ordinately applied to the diameter.

First, let the section $AGBD$ be an ellipse. Find the center c by the first article, and through it draw the diameter GL . Through the vertex L draw the straight line LF parallel to AB . Then if LF touch the ellipse, AB , LG will be conjugate diameters, by Cor. 2. Prop. IV. Book II. and LG will be ordinately applied to AB . But if LF do not touch the ellipse, let it meet the curve again in F . Draw GF , and let it meet AB in K , and GF will be the ordinate required. For, as CK , LF are parallel, (2. vi.) $GC : GK :: CL : KF$, and therefore (14. v.) GF is bisected in K . Secondly, let the section FBG be an hyperbola, or parabola. In the diameter AB take any point M , and the straight line GM being drawn, produce it to L , so that ML may be equal to GM . Then if the point L be in the curve, GL will be the ordinate required; but if L be not in the curve, draw the straight line LF parallel to AB , and let it meet the curve in F , according to Prop. IX. Book I. or Cor. 1. Def. I. of this. Draw GF , and it will be the ordinate required. For, if it meet AB in K , it may be proved, as above, that GF is bisected in K . Fig. 109.

4. The curve of a conic section FBG being given, and a point B being given in it, let it be required to draw a straight line through B to touch the section. Fig. 110.

BOOK III. Through B draw AB , a diameter of the section, and, by the last article, draw GF an ordinate to it. Through the vertex B draw the straight line TB parallel to GF , and, by Prop. II. TB will be the tangent required.

Fig. 111. 5. Two unequal straight lines AB , DE being given, bisecting one another in C at right angles, let it be required to describe the curve of an ellipse, of which AB , DE shall be the axes, and C the center.

Let AB be greater than DE , and consequently the transverse axis. Find the foci F , O in AB , by Cor. 2. Def. XI. Book II. Let the ends of a thread or string, equal in length to AB , be fixed to the points F , O . By means of the pin P let the thread or string be stretched; and, while it continues uniformly tense, let the end or point of the pin P move round in the plane, in which AB , DE are situated, till it return to the same place from which it began to move. The line traced by the end or point of the pin P is the curve of an ellipse, as is evident from Prop. XIII. Book II. and AB , DE are the axes.

Fig. 112. 6. Two straight lines AB , DE being given, bisecting one another in C at right angles, let it be required to describe the curve of an hyperbola, of which AB shall be the transverse and DE the conjugate axes.

In AB produced both ways let the foci F , O be found, by Cor. 3. Def. XI. Book II. Let one end of a thread or string $FP R$ be fixed to the point F , and let the other end be fixed to the extremity R of the ruler OR , and let the length of the ruler exceed the length of the thread or string by the straight line AB . Let O , the other extremity of the ruler, be fixed to the point O , and let the ruler revolve about O as a center. By means of the pin P let the thread or string be stretched, and let the part between P and R be kept close to the edge of the ruler; and while the ruler revolves,

in the plane in which $A B$, $D E$ are situated. The line $G B H$ will be the curve of an hyperbola, of which $A B$ is the transverse and $D E$ is the conjugate axes, and C is the center, as is evident from Prop. XIII. Book II.

7. The straight line $D x$, of indefinite length, being given, and F being a point given without it, let it be required to describe the curve of a parabola, of which $D x$ shall be the directrix and F the focus.

Fig. 113.

Place the edge of a ruler $R D x L$ along the line $D x$, and keep it fixed in that position. Let $G y E$ be a ruler of such a form that the part $G y$ may slide along the edge $D x$ of the fixed ruler $R D x L$, and the part $y E$ may be always perpendicular to $D x$. Let $E F F$ be a thread or string of the same length with the part $y E$ of the moving ruler, and let one end of it be fixed to the ruler at E , and let the other end be fixed to the point F . By means of the pin F let the thread or string be stretched, and the part between F and E be kept close to the edge of the ruler. While the ruler $G y E$ slides along the edge $D x$ of the fixed ruler, and the thread or string is kept uniformly tense, let the end or point of the pin F trace the line $A P B C$ on the plane, in which the line $D x$ and the point F are situated. The line $A P B C$ will be the curve of a parabola, of which $D x$ is the directrix and F the focus, as is evident from Cor. 3. Prop. VIII.

Several writers on Conic Sections have defined the ellipse, hyperbola, and parabola by the description in Article 5, 6, and 7 respectively; and from these descriptions, as founded on a primary, they have deduced other properties of the sections.

8. Two straight lines $A B$, $D E$ being given, bisecting one another in C but not at right angles, let it be

Fig. 114.

115.

re-



BOOK required to describe an ellipse, or hyperbola, of which
III. AB, DE shall be conjugate diameters, and c the center.

In CD , produced in the ellipse but between c and D in the hyperbola, take the point N , so that the rectangle under CD, DN may be equal to the square of CB . Through D draw the straight line Ma parallel to AB , and bisect CN in I . Draw IP perpendicular to CN , and let it meet Ma in P ; and then it is evident (4. i.) that straight lines drawn from P to N and c will be equal. With P as a center therefore, and PN or PC as a distance, let the circle MCQ be described, and let it meet the straight line Ma in M and Q . Draw the straight lines QC, MC ; and from D draw DK perpendicular to QC , and DH perpendicular to MC . In QC take CL, CR each a mean proportional between CQ, CK ; and in MC take CF and CG each a mean proportional between CM, CH . Then will KL, GF be the axes of the ellipse, or hyperbola, proposed to be described, as is evident from Cor. 2. Prop. IX. (and 31. iii.) and Prop. VII. Book II. Consequently the foci may be found, and the descriptions of the curves may be completed, as in the 5th and 6th Articles.

Fig. 116. 9. The straight line GR being given in position and magnitude, and the straight line AB bisecting it in B , let it be required to describe a parabola, of which AB shall be a diameter, and GE a double ordinate to it. Let the straight line P be a third proportional to AB, BG , and produce BA to Y , so that AY may be a fourth part of P . Through Y draw DX at right angles to YB , and through A draw AN parallel to GE . Make the angle NAF equal to the angle NAY , and make AF equal to AY . A parabola described with the focus F and the directrix DX , as in the 7th Article, will be the section required, as is evident from Prop. II. IX. and Cor. 2. and 3. Prop. XI.

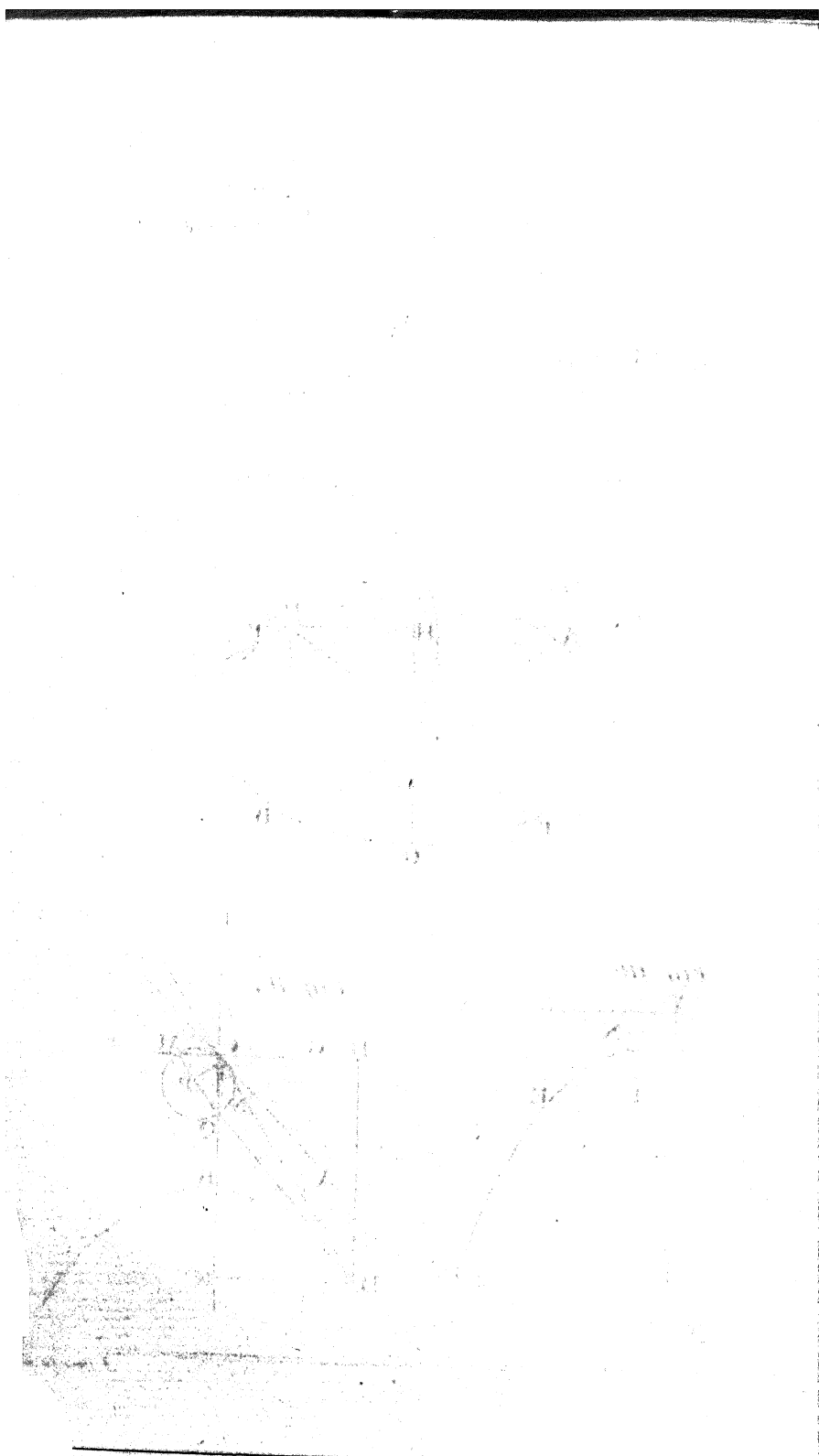
10. The

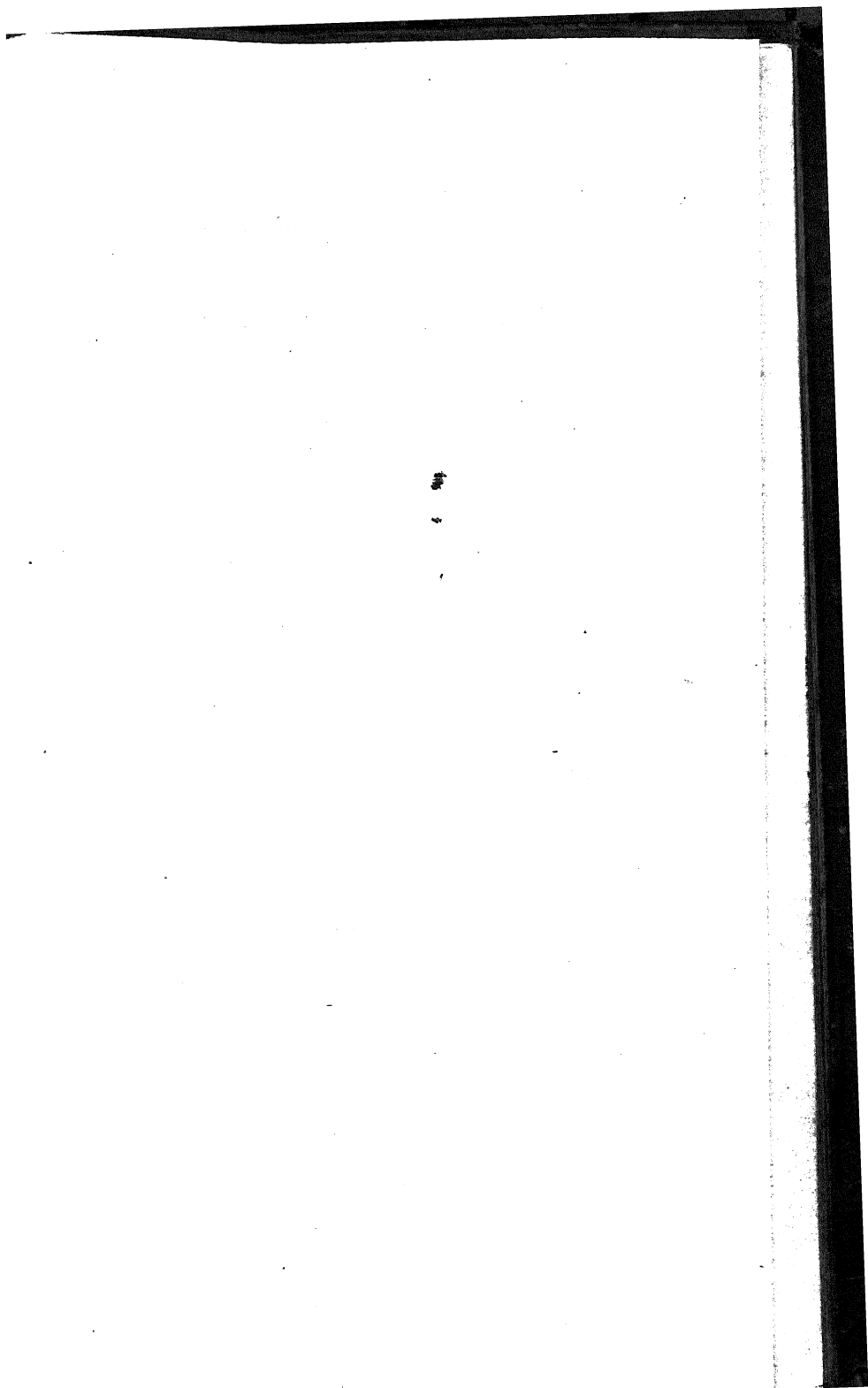
to find the diameter conjugate to AB .

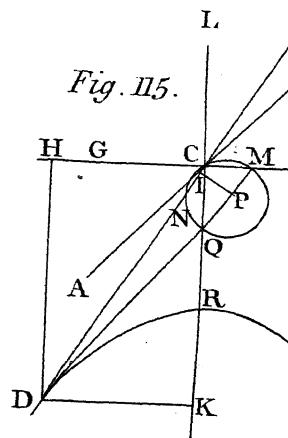
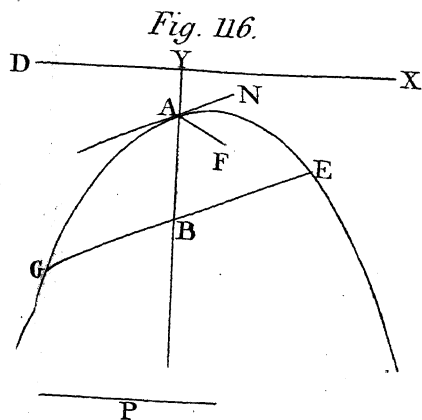
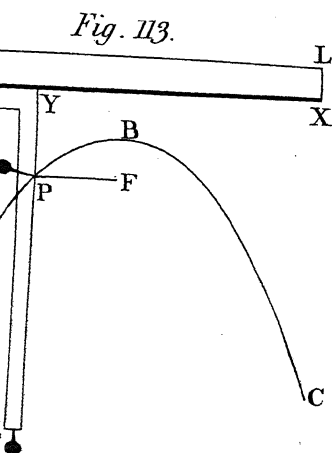
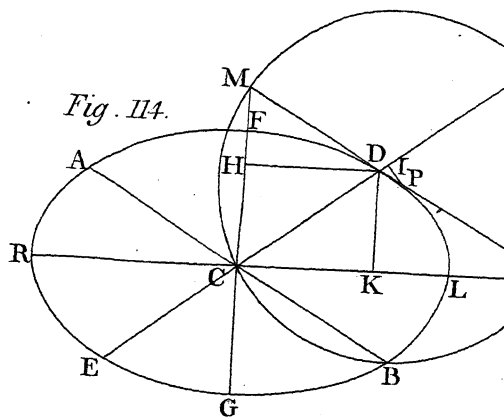
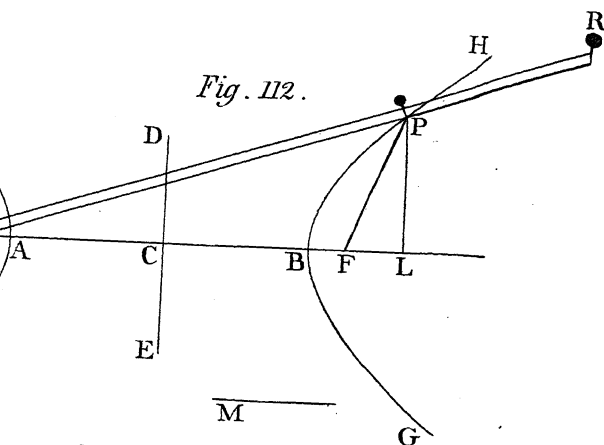
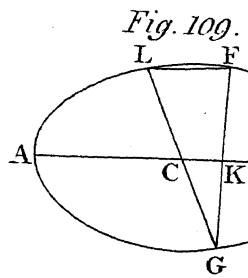
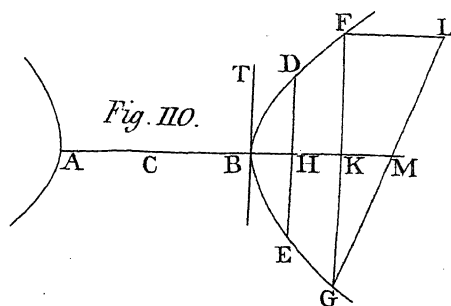
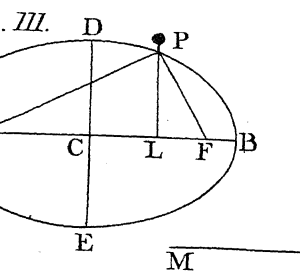
Fig. 111.
112.

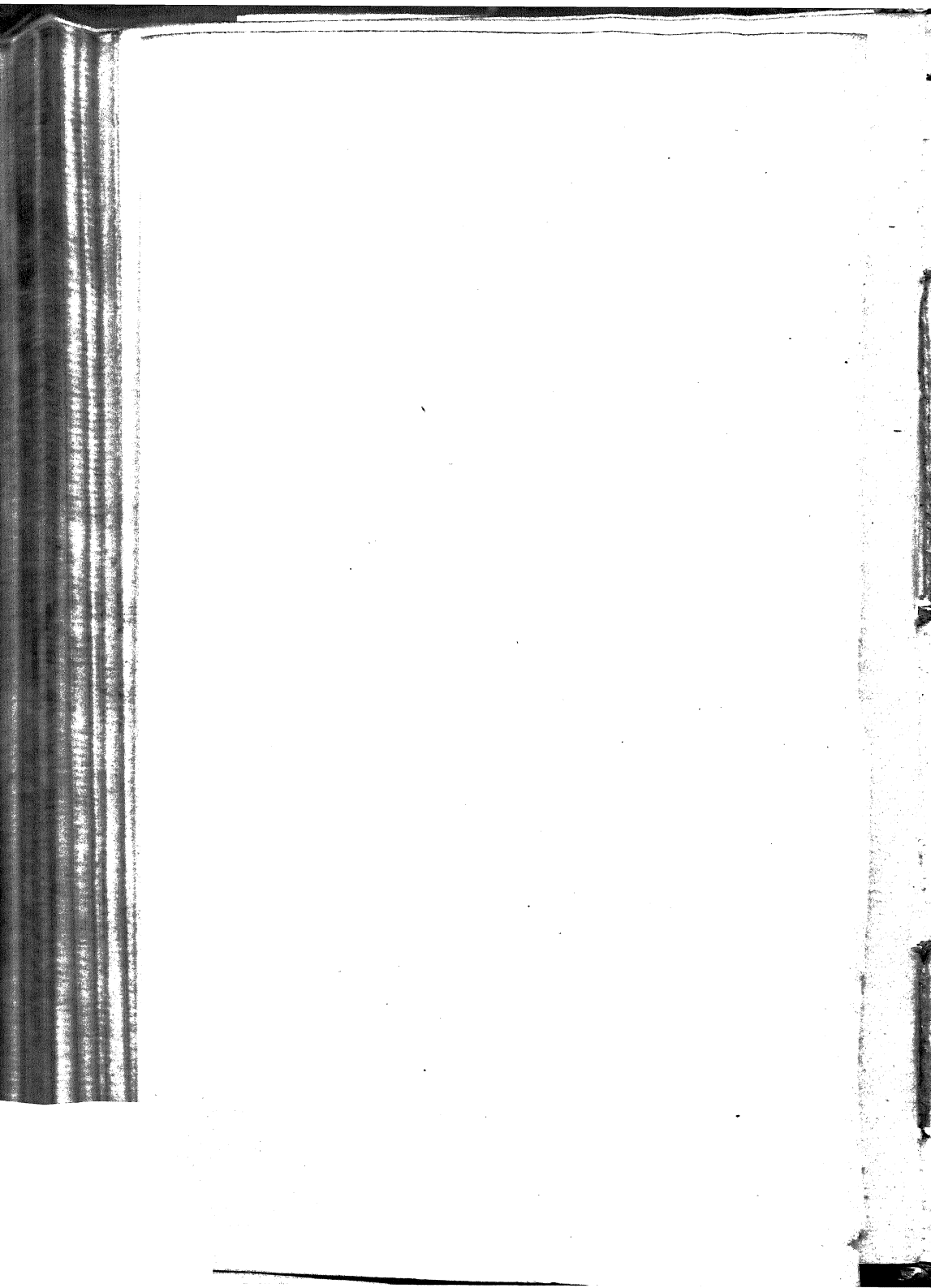
Let the straight line M be a mean proportional (13. vi.) between the abscisses AF , FB ; and, c being the center of the section, let M be to PL as CB to CD a straight line parallel to PL . Then will CD be half the conjugate diameter required. For, by hypothesis, (and 22. vi.) $M^2 : PL^2 :: CB^2 : CD^2$. But (17. vi.) $M^2 = AL \times LB$, and therefore $AL \times LB : PL^2 :: CB^2 : CD^2$. Consequently CD is the semiconjugate diameter to AB , by Prop. IV. and V. and Def. VIII. Book II.











LEMMAS
FOR
THE FOURTH BOOK
OF
CONIC SECTIONS.

LEMMA IX.*

*Let it be required to draw a straight line to touch two
given circles A L B, D E M.*

Let c be the center of $A L B$, and F the center of $D E M$, and draw $c F$. Let $c G$ be the excess of the radius of $A L B$ above the radius of $D E M$; and with c as a center, and $c G$ as a distance, describe the circle $H G$. From the point F draw $F H$ (17. iii.) touching the circle $H G$ in H . Draw $c H$, and let it meet the circumference of $A L B$ in A ; and draw $F D$ parallel to $c A$, and let it meet the circumference of $D E M$ in D . Draw $A D$, and it will touch the given circles. For by the construction $A H$, $D F$ are equal and parallel, and (16. iii.) $A H F$ is a right angle. Consequently (33. i.) $A D F H$ is a parallelogram, and (Cor. 46. i.)

* The numbering of the Lemmas is continued from those prefixed to the first Book. This manner of numbering them was found most convenient for reference.

the

the angles at A and D are right ones, and therefore (16. iii.) AD touches the circles.

If it be required to draw a straight line to touch the circles and cut CF, let CF be so divided in K that CK may be to KF as the radius of the circle ALB to the radius of the circle DEM. Draw KL (17. iii.) to touch the circle ALB in L. Draw CL and FM parallel to it, and let FM meet LK in M; and LK will touch the circle DEM in M. For (15. and 29. i.) the triangles CKL, FKM are equiangular, and therefore (4. vi.) $CK : KF :: CL : FM$. Consequently, by the construction, the point M is in the circumference of the circle DEM, and as the angles FMK, CLK are equal, and the angle CLK a right one, LM (16. iii.) touches the circle DEM in M.

LEMMA X.

If a magnitude A be to a magnitude B as a magnitude C to a magnitude D; then A will be to B as the difference of the antecedents A, C to the difference of the consequents B, D.

For let A be greater than C, and consequently (14. v.) B greater than D. Then, by alternation, $A : C :: B : D$, and (17. v.) $A - C : C :: B - D : D$; and again, by alternation, $C : D :: A - C : B - D$. But, by hypothesis, $A : B :: C : D$, and therefore (11. v.) $A : B :: A - C : B - D$.

DEFINITIONS.

I.

Fig. 129.
130. If the straight line AD be so divided in the points n, c, that the whole line AD be to one of the extreme parts AB, as the other extreme part DC to the middle part BC, the straight line AD is said to be *Harmonically*

monically divided; and the points A, B, C, D are called the *Points of harmonical division*, or *Harmonical Points*.

Cor. 1. It is evident that each extreme part is greater than the middle part.

Cor. 2. If the two extreme points A, D, and B one of the middle points of an harmonical division be given, the other point c may be found (10. vi.) by dividing the segment BD in c, so that the part CD may be to BC as AD to AB. It is evident that no other point besides c can be found, which can be a fourth point of this division.

Cor. 3. The two middle points B, c, and A one of the extreme points of an harmonical division being given, the other point D may be found. For from the point A draw the straight line AE, and as AB to BC, so let AE be to the segment EF, taken towards A. Draw FC, and let ED drawn parallel to FC meet AC produced in D. Then (2. vi.) $AD : DC :: AE : EF$; and therefore by the construction $AD : DC :: AB : BC$, and consequently D is the other point of the division. It is evident that no other point besides D can be found, which can be a fourth point of this division. For, by conversion, as the excess of AB above BC to BC, so is AC to CD.

II.

If the straight line ad be divided harmonically in the points a, b, c, d , and four straight lines aE, bE, cE, dE , any way produced through the points of division, be parallel or meet one another in the point E; these four straight lines are called *Harmonicals*. Fig. 129.
130.

Cor. Every thing remaining as above, any straight line AD, parallel to ad , and meeting the harmonicals in A, B, C, D, will be harmonically divided in these points. For, (29. and 15. i.) on account of the equiangular triangles,

angles, $ad : de :: AD : DE$, and $de : dc :: DE : DC$. Consequently,

$$ad : de : de$$

$$AD : DE : DC,$$

and (22. v.) $ad : dc :: AD : DC$. Again $ab : be :: AB : BE$, and $be : bc :: BE : BC$; and therefore

$$ab : be : bc$$

$$AB : BE : BC.$$

Consequently (22. v.) $ab : bc :: AB : BC$; and therefore (11. v.) $AD : DC :: AB : BC$, as, by hypothesis, $ad : dc :: ab : bc$.

LEMMA XI.

Fig. 129. *The rest remaining as in the first and second Definitions and their Corollaries, if a straight line GH parallel to any one of the harmonicals aE, bE, cE, dE, meet the other three, it will be bisected in the middle point of concurrence. And, on the contrary, if four straight lines ED, EC, EB, EA meet one another in E, and if the straight line GH parallel to any one of them, and meeting the other three, be bisected in the middle point of concurrence, the straight lines ED, EC, EB, EA will be harmonicals.*

Fig. 130. Part I. First, let the straight line GH be parallel to the harmonical dE, and meet aE, bE, cE in G, B, H; GH is bisected in the middle point B. For through B draw the straight line AD parallel to ad, and meeting the harmonicals aE, cE, dE in A, C, D. Then the straight line AD is harmonically divided, by the Cor. to the second Definition, and therefore $DA : AB :: DC : CB$. But (4. vi.) $DA : AB :: DE : BG$; and $DC : CB :: DE : BH$. Consequently (11. v.) $DE : BG :: DE : BH$, and therefore (14. v.) GB, BH are equal.

Se-

Secondly, let the straight line GH be parallel to the harmonical CE , and meet AE, BE, DE in A, H, G ; it will be bisected in the middle point A . For through A draw the straight line AD parallel to ad , and meeting the harmonicals BE, CE, DE in B, C, D . Then, by the Cor. to the second Definition, $AD : DC :: AB : C$. But (4. vi.) $AD : DC :: GA : EC$; and $AB : C :: AH : EC$. Consequently (II. v.) $GA : EC :: H : EC$, and therefore (14. v.) GA, AH are equal.

Fig. 129.

Part II. First, let the straight line GH be parallel to ED , and meet the straight lines EC, EB, EA in H, B , and let it be bisected in the middle point A . Through A draw the straight line AD , and let it meet the straight lines EA, EC, ED in A, C, D . Then (4. vi.) $DE : BG :: A : AB$, and $DE : BH :: DC : CB$. Consequently (II. v.) $DA : AB :: DC : CB$.

Fig. 130.

Secondly, let the straight line GH be parallel to EC , and meet the straight lines ED, EB, EA in the points H, A , and let it be bisected in the middle point A . Through A draw the straight line AD , meeting the straight lines EB, EC, ED in B, C, D . Then (4. vi.) $A : EC :: AD : DC$; and $AH : EC :: AB : BC$. Consequently (7. and II. v.) $AD : DC :: AB : BC$.

Fig. 129.

LEMMA XII.

four harmonicals meet any straight line, the straight line will be harmonically divided in the points of concurrence.

If the harmonicals are parallel to one another, this is evident (from 10. vi.), but if not, let AE, BE, CE, DE be

Fig. 130.

four harmonicals, and let them meet any straight line AD in A, B, C, D . Through B draw the straight line GH parallel to ED , and let it meet the straight lines EA in G , and EC in H . Then, by the preceding Lemma,

Lemma, GH is bisected in B; and (4. vi.) $DA : AB :: DE : BG$ or BH ; and $DE : BH :: DC : CB$. Consequently (II. v.) $DA : AB :: DC : CB$.

A
GEOMETRICAL TREATISE
OF
CONIC SECTIONS.

BOOK IV.

Of similar Sections, general Properties, Circles having the same curvature with the Sections in given points, and of straight lines cut harmonically by the Sections. This Book also contains Problems useful in the Theory of Astronomy, and Methods of finding two mean proportionals and of trisectioning an angle, by means of the Sections.

DEFINITIONS.

I.

TWO segments of conic sections are called *Similar Segments*, if a rectilineal figure can be inscribed in one of them similar to any rectilineal figure inscribed in the other.

II.

Two conic sections are called *Similar Sections*, if a rectilineal figure can be inscribed in one of them similar

BOOK lar to any rectilineal figure inscribed in the other : and
IV. two conic sections are also called *Similar*, if a segment can be taken of one of them similar to a segment of the other.

III.

If a straight line touch two conic sections in the same point, the two sections are said to *touch one another* in the same point.

IV.

If a circle so touch a conic section in any point, that no other circle, touching it in the same point, can pass between it and the section, on either side of the point of contact, it is said to have the *same Curvature with the Section in the point of contact*; or it is said to be the *Osculating Circle for that point*.

PROP. I.

Any two parabolas are similar to one another ; and the similar rectilineal figures inscribed in them are to one another as the squares of the parameters of the axes.

Fig. 117.
118.

Let ABC , abc be two parabolas, of which BE , be are the axes, and p , p their parameters ; and let $ABCD$ be any rectilineal figure inscribed in the one parabola ; a rectilineal figure similar to $ABCD$ may be inscribed in the other, and the similar rectilineal figures inscribed in them are to one another as p^2 to p^2 .

For draw from b , the vertex, the straight line ba to the curve, so that the angle $e b a$ may be equal to the angle $E B A$. Draw bc to the curve, so that the angle $e b c$ may be equal to the angle $E B C$. Draw BD , and draw bd to the curve, so that the angle $e b d$ may be equal to the angle $E B D$; and draw ad . Let AE be an ordinate to the axis BE , and let ae be an ordinate to the axis be . Then, as the angles at E and e are right

right angles, the triangles ABE , abe are equiangular, and (4. vi.) $BE : EA :: be : ea$. But, by the sixth Definition Book III. $BE : EA :: EA : P$; and $be : ea :: ea : p$. We have therefore the two following ranks of magnitudes, in which the magnitudes taken two and two in the same order have the same ratio to one another;

$$BE : EA : P$$

$$be : ea : p$$

and consequently (22. v.) $BE : P :: be : p$; or by alternation $BE : be :: P : p$. But (4. vi. and altern.) $BE : be :: BA : ba$, and therefore (11. v.) $BA : ba :: P : p$. In the same way it may be proved, that $BC : bc :: P : p$, and that $BD : bd :: P : p$; and therefore (11. v.) $BC : bc :: BD : bd$, and by alternation $BC : BD :: bc : bd$. But as the angles EBD , ebd are equal, and the angle EBC equal to the angle ebc , the angles DBC , dbc are equal. Consequently (6. vi.) the triangles CBD , cbd are equiangular, and (4. vi. and altern.) $CD : cd :: BC : bc$, or P to p . In the same way it may be proved that $AD : ad :: P : p$; and therefore the rectilinear figure $abcd$ is similar to the rectilinear figure $ABCD$. The parabolas ABC , abc are therefore similar according to the first and second Definitions, and it is evident (Cor. 2. 20. vi.) that the rectilinear figures $ABCD$, $abcd$ are to one another as the squares of their homologous sides, or (11. v.) as P^2 to p^2 .

Cor. 1. It is evident from the above that similar rectilinear figures may be inscribed in the similar parabolic segments $ABCD$, $abcd$, of which the homologous sides will be to one another as P to p , and which will be deficient from the parabolic segments by spaces less than any given. The similar parabolic segments themselves will therefore be to one another as P^2 to p^2 .

M

Cor.

BOOK
IV.

Cor. 2. In two parabolas the parameters of diameters, which contain equal angles with their ordinates, are to one another as the parameters of the axes. For, the rest remaining as above, let AG , touching the parabola ABC in A , meet the axis BE in G ; and let ag , touching the parabola abc in a , meet the axis be in g . Let F be the focus of ABC , and f the focus of abc ; and draw AF , af . Then, as BF is one fourth of p , and bf one fourth of p , by the above, (and 15. v.) $AB : BF :: ab : bf$; and therefore (6. vi.) the triangles ABF , abf are equiangular, and $AF : BF :: af : bf$; or, by Cor. 2. Prop. XI. Book III. (and 15. v.) the parameter of the diameter passing through A is to p as the parameter of the diameter passing through a to p . Again, by Cor. Prop. IX. Book III. the triangles AFG , afg are isosceles, and, as above, the angles AFG , afg are equal. The angles AGF , agf are therefore equal; and, as the diameters of a parabola are parallel, the angle AGF is equal to the angle which the diameter passing through A contains with its ordinates, and the angle agf equal to the angle which the diameter passing through a contains with its ordinates, by Prop. II. Book III.

PROP. II.

Two ellipses, or two hyperbolas, are similar to one another, if two conjugate diameters in the one be proportional to two conjugate diameters in the other, and the first two and the other two contain equal angles. On the contrary, if two ellipses, or two hyperbolas, be similar to one another, two conjugate diameters in the one will be proportional to two conjugate diameters in the other, provided the first two and the other two contain equal angles.

Let

Let ВН , bb be two ellipses, or two hyperbolas, and in the one let АВ a diameter be to DE its conjugate as, in the other, ab a diameter is to de its conjugate, and, c, c being the centers, let the angle DCB be equal to the angle $dc b$; then the ellipse ВН is similar to the ellipse bb , and the hyperbola ВН is similar to the hyperbola bb . On the contrary, if the ellipse or hyperbola ВН be similar to the ellipse or hyperbola bb , and if the angle DCB contained by AB , DE , two conjugate diameters in the one, be equal to the angle $dc b$ contained by ab, de , two conjugate diameters in the other; then AB is to DE as ab to de .

BOOK
IV.Fig. 119.
120.
121.
122.

Part I. Let ВНМЛК be any rectilineal figure inscribed in the ellipse or hyperbola ВН , and from c the center draw the straight lines ch, cm, cl, ck . Again, at c the center of the ellipse or hyperbola bb , make the angles bcb, bcm, mcl, lck , equal to the angles ВСН , НСМ , МСЛ , ЛСК , each to each; and let the points b, m, l, k be in the curve of the section. Then the straight lines bb, bm, ml, lk, kb being drawn, the rectilineal figure $bbmlk$ inscribed in the ellipse bb is similar to ВНМЛК inscribed in the ellipse ВН ; and the rectilineal figure $bbmlk$ inscribed in the hyperbola bb is similar to ВНМЛК inscribed in the hyperbola ВН . For draw HG an ordinate to AB , and hg an ordinate to ab . Then, as the angles DCG , dcg are equal, and as, by Cor. 2. Prop. IV. Book II. HG is parallel to DC , and hg parallel to dc , the angles (29. I.) at G and g are equal. The angles HCG , bcg are also equal, by construction, and therefore the triangles HCG , bcg are equiangular. Consequently (4. vi.) $\text{GH} : \text{CG} :: gb : cg$; and $\text{GH}^2 : \text{CG}^2 :: gb^2 : cg^2$, and (16. v.) $\text{GH}^2 : gb^2 :: \text{CG}^2 : cg^2$. But, by hypothesis, (and 15. v.) $\text{CB} : \text{CD} :: cb : cd$, and therefore $\text{CB}^2 : \text{CD}^2 :: cb^2 : cd^2$; and,

M 2

by

BOOK by Prop. V. Book II. $CB^2 : CD^2 :: AG \times GB : GH^2$;
 IV. and $cb^2 : cd^2 :: ag \times gb : gb^2$. Hence (11. v.) $AG \times GB : GH^2 :: ag \times gb : gb^2$; and (16. v.) $AG \times GB : ag \times gb :: GH^2 : gb^2$. Consequently (11. v.) $AG \times GB : ag \times gb :: CG^2 : cg^2$; and therefore (16. v.) $AG \times GB : CG^2 :: ag \times gb : cg^2$, and (by 5. and 6. ii. and 17. and 18. v.) $CB^2 : CG^2 :: cb^2 : cg^2$. We have therefore (22. vi.) $CB : CG :: cb : cg$; and (16. v.) $CB : cb :: CG : cg$. Again, by the similar triangles, $CG : cg :: CH : cb$; and therefore (11. v.) $CB : cb :: CH : cb$, and by alternation $CB : CH :: cb : cb$. Consequently (6. vi.) the triangles BCH , bcb are similar. In the same way it may be proved that $CB : cb :: CM : cm$; and therefore (11. v.) that $CH : cb :: CM : cm$. Consequently, by alternation, (and 6. vi.) the triangles MCH , mcb are similar; and in the same way it may be proved, that the triangle lcm is similar to the triangle LCM , the triangle lck to the triangle LCK , and the triangle kcb to the triangle KCB . The rectilineal figures $BHMLK$, $b b m l k$ (20. vi.) are therefore similar, and the sections BH , bb are therefore similar, according to the first and second Definitions.

Part II. If ab be not to dc as AB to DE , let AB be to DE as ab to a straight line greater or less than dc ; and with this straight line as a conjugate diameter, and AB as a transverse diameter, suppose an ellipse or hyperbola to be described. Then this ellipse or hyperbola will fall without or within the section bb of the same name, and, by the preceding part, a rectilineal figure may be inscribed in the section, at present supposed to be described, similar to the rectilineal figure $BHMLK$. But as, by hypothesis, the sections BH , bb are similar, a rectilineal figure may be inscribed in the section bb similar to the rectilineal figure $BHMLK$,

according to the first and second Definitions. Let this inscribed figure be $bbmlk$. Then (21. vi.) in the section, having ab for its transverse diameter, and falling either without or within the section bb , a rectilinear figure may be inscribed similar to $bbmlk$; which (from Def. i. vi. and 21. i.) is evidently absurd. Consequently, the sections BH , bb being similar, and the angles DCB , dcb equal, AB is to DE as ab to dc .

BOOK
IV.

Cor. 1. In similar ellipses, or similar hyperbolas, diameters which contain equal angles with the axes are to one another as the axes. For if AB , ab be the transverse, and DE , de the conjugate axes, then the angles BCB , bcb being equal, CB is to CH as cb to ch , as was above demonstrated.

Cor. 2. From the above (and Cor. 2. 20. vi.) it is evident that similar rectilinear figures may be inscribed in similar ellipses, or in similar hyperbolic segments, which shall be to one another as the squares of the transverse, or as the squares of the conjugate axes.

Cor. 3. Similar ellipses, and similar hyperbolic segments, are to one another as the squares of their transverse, or as the squares of their conjugate axes. This is evident from the preceding Cor. (and 2. xii.) as a rectilinear figure may be inscribed in an ellipse, or in a hyperbolic segment, which shall be deficient from the ellipse, or hyperbolic segment, by a space less than any given space.

Cor. 4. The angles contained by the asymptotes of similar hyperbolas are equal to one another; and if the angles contained by the asymptotes of two hyperbolas be equal, the hyperbolas will be similar. For let AB , DE be the axes, c the center, and cs an asymptote of the hyperbola BH ; and let ab , de be the axes, c the center, and cs an asymptote of the hyperbola bb . Let BS touch the hyperbola BH in the vertex B , and meet

BOOK IV. the asymptote in s ; and let bs touch the hyperbola bb in the vertex b , and meet the asymptote in s . Then the angle cbs is equal to the angle cbs , as each is a right one, and, by Cor. 3. Prop. XV. Book III. bs is equal to ce , and bs is equal to ce ; and therefore, if the hyperbolas be similar, by the second part of this Prop. (and 15. v.) $cb : bs :: cb : bs$. Consequently (6. vi.) the angle bcs is equal to the angle bcs . On the contrary, if the angle bcs be equal to the angle bcs , then (4. vi.) $cb : bs :: cb : bs$; and therefore, by the first part of this Prop. the hyperbolas bh , bb are similar.

PROP. III.

If two ellipses, or two hyperbolas, be similar, the transverse axis in the one will be to the distance between the foci as the transverse axis in the other to the distance between the foci. On the contrary, two ellipses, or two hyperbolas, will be similar, if the transverse axis in the one be to the distance between the foci as the transverse axis in the other to the distance between the foci.

Fig. 119. Part I. Let bh , bb be two similar ellipses, and bh ,
120. bb be two similar hyperbolas. Let ab , de be the
121. transverse and conjugate axes, c the center, and f , o
122. the foci of the one; and let ab , de be the transverse and conjugate axes, c the center, and f , o the foci of the other; then ab is to fo as ab is to fo .

For in the ellipses draw eo , eo , but in the hyperbolas draw ea , ea . Then, by Cor. 2. and Cor. 3. Def. XI. Book II. in the ellipses eo is equal to ca . and eo is equal to ca ; but in the hyperbolas ea is equal to co , and ea is equal to co . Hence, as the section bh is similar to the section bb , by the second part of Prop. II. in the ellipses $eo : ec :: eo : ec$; but

but in the hyperbolas $CA : CE :: ca : ce$. Consequently, as the angles at c and c are equal, being right angles, the triangles (7. vi.) ECO, eco in the ellipses are equiangular; and in the hyperbolas the triangles (6. vi.) ACE, ace are equiangular. Hence in the ellipses EO , or its equal CA , is to CO as EO , or its equal ca , is to co ; but in the hyperbolas CA is to AE , or its equal CO , as ca is to ae , or its equal co .

Part II. On the contrary, the rest remaining as above, if it be allowed, either in the ellipses or hyperbolas, that CA is to CO as ca to co , then it may be proved, as above, (by 7. vi.) that in the ellipses the triangles ECO, eco are equiangular, but in the hyperbolas that the triangles ACE, ace are equiangular. Consequently in the ellipses EO , or its equal CA , is to CE as EO , or its equal ca , to ce ; and in the hyperbolas CA is to CE as ca to ce . Hence this part of the Cor. is evident from the first part of Prop. II.

Cor. If BH, bh be two ellipses, or two hyperbolas, of which the foci are O, F in the one, and o, f in the other, and if the triangles MOF, mof be equiangular; then if the sections be similar, and the point M be in the curve of BH , the point m will be in the curve of bh . For (4. vi.) $OM : MF :: om : mf$; and therefore (18. v.) in the ellipses $OM + MF : MF :: om + mf : mf$; and (17. v.) in the hyperbolas $OM - MF : MF :: om - mf : mf$. Again, (4. vi.) either in the ellipses or hyperbolas, $MF : FO :: mf : fo$. Consequently, in the ellipses,

$$OM + MF : MF : FO$$

$$om + mf : mf : fo;$$

and therefore (22. v.) $OM + MF : FO :: om + mf : fo$.

Also, in the hyperbolas,

$$OM - MF : MF : FO$$

$$om - mf : mf : fo;$$

BOOK and therefore (22. v.) $OM - MF : FO :: om - mf :$
 IV. fo . But, the rest remaining as in the Proposition, by
 Prop. XIII. Book II. $OM + MF$ in the ellipse, but
 $OM - MF$ in the hyperbola, is equal to AB , and as
 the sections are similar, by this Prop. $AB : FO :: ab :$
 fo . Consequently (11. v.) in the ellipse $ab : fo ::$
 $om + mf : fo$; and in the hyperbola $ab : fo :: om -$
 $mf : fo$. In the ellipse, therefore, (14. v.) $om + mf$ is
 equal to ab , and in the hyperbola $om - mf$ is equal to
 ab . Consequently the point m is in the curve of the
 section bb , by Cor. 1. Prop. XIV. Book II.

SCHOLIUM.

Having explained methods for describing conic sections, and demonstrated the principal properties of similar sections, it may be proper, in this place, to give an explanation of some passages in the fourth section of the first Book of the Principia. The reader, who thinks such explanations necessary, is supposed to have that justly celebrated work before him whilst he peruses this Scholium, as he is here referred to the figures which it contains. In the following explanations the Lemmas and Propositions of the above-mentioned Section of the Principia are printed in capital letters, to distinguish them from such parts of this treatise as are referred to.

LEMMA XV. This is fully explained in Cor. 2. Prop. XV. Book II.

PROPOSITION XVIII. This is evident from Prop. XIII. and Cor. 2. Prop. XV. Book II.

PROPOSITION XIX. The reasons for the descriptions here delivered are easily deduced from Cor. 3. Prop. X. and Cor. 3. Prop. VIII. Book III. and Lemma IX. and it is evident that FI will be the directrix of the parabola.

PROPO-

PROPOSITION XX. An ellipse or hyperbola is said to be given in species when it is similar to a given ellipse or hyperbola, or when the ratio of the axes to one another is given. When the ratio of the axes to one another is given, the ratio of the principal or transverse axis to the distance of the foci is also given, or is constant, according to Prop. III.

BOOK
IV.

The remaining part of Case 1. PROP. XX. is evident from Cor. 1. Prop. VIII. Book III. and Lemma IX.

Case 2. That a straight line bisecting vv at right angles will pass through the other focus, of the ellipse or hyperbola, has been proved in Cor. 3. Prop. XV. Book II. and that the circumference of a circle described upon κk as a diameter will pass through the same focus has been proved in the 4th Cor. to the same Proposition. The whole of Case 2. is therefore evident from the Corollaries already mentioned, and from Prop. XIII. and XV. Book II.

Case 3. That a circle described upon κk as a diameter will pass through the other focus is evident from Cor. 4. Prop. XV. Book II. and that vr will pass through the same focus is evident from Prop. XV. Book II.

Case 4. That vb is equal to ab is evident from Sir Isaac Newton's demonstration. Recurring to proportions previously stated $sh : sb :: sp : sp$, and therefore $sh : sp :: sb : sp$; and as the angles $ps h, p s b$ are equal, the triangles $ps h, p s b$ are similar. Again $sv : sp :: sb : sq$; and on account of the similar triangles $v s p, b s q$, $sv : sp :: bs : sq$. Consequently (II. v.) $sv : sp :: sv : sp$, and by inversion $sp : sv :: sp : sv$. From the above therefore

$$sh : sp : sv$$

$$sb : sp : sv,$$

and

BOOK IV. and (22. v.) $SH : SV :: sb : sv$, and as the angles $VS H, vsb$ are equal, the triangles (6. vi.) $VS H, vsb$ are similar. From hence, and Prop. III. and its Cor. the proceedings in this case are evidently just.

LEMMA XVI. Case 1. By Cor. 1. Prop. VIII. Book III. PR is the directrix of the hyperbola, of which NM is the transverse axis, and A, B the foci, and therefore AP is given; and by the same Cor. $ZR : AZ :: MN : AB$. Let D denote the difference between AZ, CZ , and then D by hypothesis is given. For the same reasons AQ is given, SA being the directrix to the other hyperbola; and, as above, $AZ : ZS :: AC : D$. Consequently, by the fifth Lemma, (and I. vi.) $ZR : ZS :: MN \times AC : AB \times D$; and as MN, AC, AB , and D are given, the ratio of ZR to ZS is given. Again, $\text{Radius} : \text{fine} \angle ZTS :: TZ : ZS$; and therefore by the above, and the fifth Lemma, (and I. vi.) $\text{Radius} \times D : \text{fine} \angle ZTS \times AC :: TZ : AZ$. But TZ and TS being given in position, the fine of the angle ZTS is given, and therefore the ratio of TZ to AZ is given. The straight line TA is given, and also the angle ATZ , and $AZ : TZ :: \text{fine} \angle ATZ : \text{fine} \angle TAZ$; and therefore the angle TAZ is given. Consequently the triangle ATZ is given. By introducing some additional straight lines into the figure, the triangle might be ascertained geometrically.

The other two cases need no explanation; nor do the remaining parts of the Section, as the several cases in PROP. XXI. may be solved by the preceding LEMMA. The SCHOLIUM is evident from Prop. VIII. Book III.

PROP. IV.

If two straight lines touching a conic section, or opposite hyperbolas, meet a straight line which cuts the section,
oppe-

opposite hyperbola, or opposite hyperbolas, and is parallel to the straight line joining the points of contact, the segments of the secant between the curve or curves and the tangents will be equal to one another: and if two straight lines touching a conic section meet a straight line which touches the section, or opposite hyperbola, and is parallel to the straight line joining the points of contact, the segments of the last mentioned tangent between the point of contact and the tangents which it meets will be equal to one another.

BOOK
IV.

If the two tangents be parallel, this Prop. is the same as the Cor. to Prop. I. Book II. and this case has been already demonstrated. Let the two straight lines therefore AE , DE touch the conic section AD , or the opposite hyperbolas A , D , in the points A , D , and meet one another in E , and, first, let them meet in the points H , K the straight line HK , which is parallel to AD , and cuts the curve of the section, the opposite hyperbola, or the curves of the opposite hyperbolas, in the points F , G ; the segments HF , KG are equal to one another.

Fig. 131.
132.
133.
134.

For let the straight line EM bisect AD in M , and meet the straight line HK in N ; and then, by Cor. 1. to Prop. VI. Book III. EM is a diameter, and therefore, as AD , FG are parallel, FG is bisected in N , by Cor. 1. to Prop. II. Book III. But (29. i. and 4. vi.) $EM : EN :: AM : HN$, and $EM : EN :: DM : KN$. Consequently (11. v.) $AM : HN :: DM : KN$, and therefore (14. v.) HN is equal to KN , and HF equal to KG .

The rest remaining as above, let the tangents AE , DE now meet the straight line LI in the points L and I , and let LI be parallel to AD , and touch the section,

Fig. 131.
132.
134.

or

BOOK or opposite hyperbola, in the point B; the segments
IV. LB, BI are equal.

For as LI, AD are parallel, by Prop. II. Book III. a diameter passing through B will bisect AD, and, by Prop. VI. Book III. it will pass through E. The diameter EM therefore passes through B the point of contact, and (4. vi.) $EM : EB :: AM : LB$. Also $EM : EB :: DM : IB$, and therefore (II. v.) $AM : LB :: DM : IB$. Consequently (14. v.) LB is equal to IB.

Fig. 131. Cor. 1. If two parallel straight lines as AD, GF cut
132. a conic section, or opposite hyperbolas, in A, D and G,
133. F, and straight lines AG, DF joining their extremities
134. meet in O, and the straight line OQ, which cuts the section, or opposite hyperbolas, in P, R, and is parallel to AD, GF; the segments OR, QP are equal. For let the tangents AE, DE meet OQ in T and S; and then (4. vi.) $AH : AT :: HG : TO$, and $DK : DS :: KF : SQ$. But, on account of the parallels, $AH : AT :: DK : DS$, and therefore $HG : TO :: KF : SQ$; and as, by this Prop. HG is equal to KF, and TR to SP, it is evident (14. v.) that OR is equal to QP.

Fig. 131. Cor. 2. The rest remaining as above, let AG meet
132. the tangent LI in V, and let DF meet it in Y, and the
134. segments VB, YB will be equal. For it may be demonstrated, as in the preceding Cor. that LV, IY are equal; and consequently, as LB, IB are equal, the segment VB is equal to the segment YB.

PROP. V.

If a trapezium be inscribed in a conic section, or opposite hyperbolas, and its sides be indefinitely produced, and if from any point in the curve two straight lines be drawn parallel to two adjacent sides of the trapezium and meet the opposite sides; the rectangles under the segments of these straight

straight lines, between the point in the curve and the opposite sides of the trapezium, will be to one another as the squares of the semidiameters parallel to them, or the squares of the tangents parallel to them, and meeting one another.

BOOK
IV.

Let $ABDC$ be a trapezium inscribed in a conic section, or opposite hyperbolas, and let its sides be indefinitely produced, and from any point E in the curve let the straight lines EN , EH be drawn parallel to the adjacent sides AB , AC , and let EN meet the opposite sides AC , BD in Q and N , and EH meet the opposite sides AB , CD in R and H ; the rectangles QEN , HER are to one another as the squares of the semidiameters parallel to EN , EH , or the squares of the tangents parallel to them and meeting one another.

Fig. 123.
124.

For if the opposite sides AC , BD be not parallel, let the straight lines Bd , DF be drawn parallel to AC or HR ; and let Bd meet the curve again in d , and let it meet EN in n . Let DF meet the curve again in F , and the straight line AB in G . Draw the straight line cd , and let it meet the straight line HR in b , and the straight line DF in s . Let the straight line HR meet the curve again in T . Then, by Cor. 1. Prop. IV. sD , FG are equal, and bT is equal to ER , and consequently bE , TR are equal. The rectangle bER is therefore equal to the rectangle TRR ; and, on account of the parallelograms BE , AE , the rectangle QEn is equal to the rectangle ARB . The rectangles TRR , ARB are therefore to one another as the rectangles bER , QEn ; and consequently, by Prop. V. Book. II. and Prop. XIII. Book I. these rectangles are to one another as the squares of the semidiameters parallel to HR , EN , or as the squares of tangents parallel to HR , EN , and meeting one another. Again, on account of
the

BOOK the equiangular triangles HCb , DCS , $Hb : SD ::$
IV. $Cb : CS$; and therefore, on account of the equals
 and parallels, $Hb : FG :: EQ : AG$. Also, on account
 of the equiangular triangles NBN , DBG , NB or (34. i.)
 $ER : DG :: NN : GB$; and therefore, by the fifth
 Lemma, $Hb \times ER : DG \times GF :: EQ \times NN : AG \times$
 GB ; and by alternation $Hb \times ER : EQ \times NN ::$
 $DG \times GF : AG \times GB$. But, by Prop. V. Book II.
 and Prop. XIII. Book I. $DG \times GF$ and $AG \times GB$ are
 to one another as the squares of the semidiameters pa-
 rallel to HR , EN , or as the squares of tangents paral-
 lel to HR , EN , and meeting one another. By the
 above therefore (and II. v.) $bE \times ER : aE \times EN ::$
 $Hb \times ER : EQ \times NN$; and by alternation $bE \times ER :$
 $Hb \times ER :: aE \times EN : EQ \times NN$; and therefore
 (I. vi.) $bE : Hb :: EN : NN$, and (17. and 18. v.)
 $HE : Hb :: EN : NN$. Consequently by alternation
 (and I. vi.) $HE \times ER : EN \times EQ :: Hb \times ER : NN$
 $\times EQ$; and therefore, by the above, (and II. v.) HE
 $\times ER$ and $EN \times EQ$ are to one another as the squares
 of the semidiameters parallel to HR , EN , or as the
 squares of tangents parallel to HR , EN , and meeting
 one another.

Fig. 125. If the straight lines AB , CD cut one another like
 the diagonals of a trapezium, as in Fig. 125. and the
 rest be as expressed above in the particular enunciation;
 then it may be proved, as above, that $aE \times EN$ and
 $HE \times ER$ are to one another as the squares of the se-
 midiameters parallel to EN , EH , or the squares of the
 tangents parallel to them, and meeting one another.

It is evident that the method of demonstration and
 conclusion will be the same, if one of the straight lines
 HR , Bd , DF , or even two of them, be tangents.

Fig. 123.- **Cor. I.** If the points A , B , C , D remain fixed, and
 124.
 125. the point E with the straight lines EN , EH , always pa-
 rallel

QEN , HER will be to one another in the same ratio.

Cor. 2. The points A, B, C, E remaining fixed, if the point D , the intersection of the straight lines BD, CD , be carried round the section, or opposite hyperbola, in every situation of D the segments EH, EN will be to one another in the same ratio. For the rectangles HER, QEN in every situation of D will be to one another in the same ratio, and therefore as ER, QE remain fixed, the *Cor.* is (I. vi.) evident.

Cor. 3. The rest remaining as above, draw BC , and let it meet EH in L . In the straight line EQ , and on the same side of EH with the point N , let the point T be taken, and let HT be to EN as LE to ET ; and then BT being drawn, it will touch the section. For, if it be possible, let it meet the curve again in V ; and VC being drawn, let it meet EH in X . Then, by *Cor. 2.* $XE:ET::HE:EN$, and therefore (II. v.) $LE:ET::XE:ET$, and LE, XE are equal: which is absurd. The straight line BT therefore touches the section.

Cor. 4. Hence, the rest remaining, if the straight line BT touch the section in B , and meet the straight line EN in T , HE will be to EN as LE to ET .

Cor. 5. If the straight lines AR, CH touch the section in A, C , and from E , a point in the curve, EQ be drawn parallel to AR , and meet AC , the line joining the points of contact, in Q , and HER be drawn parallel to AC , and meet the tangents in R and H ; then the rectangle HER and the square of EQ will be to one another as the squares of the semidiameters parallel to HR, QE , or as the squares of the tangents parallel to HR, QE , meeting one another. For if HR meet the curve again in T , then, by *Prop. IV.* HT, ER are equal, and the rectangles HER, TRE are equal:

Fig. 126.

Fig. 127.



BOOK equal: and (34. i.) as EQ is equal to AR , the Cor. is
IV. evident from Prop. V. Book II. and Prop. XIII.
 Book I.

PROP. VI.

If a trapezium be inscribed in a conic section, or opposite hyperbolas, and from any point in the curve a straight line be drawn in a given angle to each of the sides, or the sides produced, the rectangle under the lines drawn to two opposite sides will be to the rectangle under the lines drawn to the two other opposite sides in a given ratio.

Fig. 135. Let $ABDC$ be a trapezium inscribed in the conic
 136. section $EABDC$, or in the opposite hyperbolas ADC ,
 EB , and from any point E in the curve let the straight
 lines EL , EM , EO , EK be drawn in given angles to
 the sides AB , CD , BD , AC , each to each; the rectan-
 gle under EL , EM , drawn to the opposite sides AB ,
 CD , is to the rectangle under EO , EK , drawn to the
 other two opposite sides BD , AC , in a given ratio.

For from any other point e in the curve of the sec-
 tion, or in the curve of either of the opposite hyperbo-
 las, let the straight lines el , em , eo , ek be drawn pa-
 rallel to EL , EM , EO , EK , each to each, and to the
 same side of the trapezium each to each. Through
 the point E let the straight lines AN , HR be drawn
 parallel to the adjacent sides AB , AC , and meeting the
 sides AC , BD , AB , DC , in the points A , N and R , H .
 Through the point e draw the straight lines qn , br pa-
 rallel to the straight lines AN , HR , or to the sides AB ,
 AC , and meeting the sides AC , BD , AB , DC in the
 points q , n and r , b . Then (29. i. and 4. vi.) $ER :$
 $er :: EL : el$; and $EH : eb :: EM : em$. Conse-
 quently, by the fifth Lemma, $ER \times EH : er \times eb ::$

$EL \times EM : el \times em$. Again (29. i. and 4. vi.) $EQ : EQ : EK : ek$; and $EN : en :: EO : eo$, and therefore, as before, $EQ \times EN : eq \times en :: EK \times EO : ek \times eo$. But, by Cor. I. Prop. V. $ER \times EH : er \times eb :: EQ \times EN : eq \times en$; and therefore (II. v.) $EL \times EM : el \times em :: EK \times EO : ek \times eo$. Consequently, by alternation, the rectangle under EL, EM is to the rectangle under EK, EO in the constant or given ratio of the rectangle under el, em to the rectangle under ek, eo .

Cor. If two straight lines AR, CH touch a conic section in A, C , and if from any point E in the curve straight lines EL, EM, EK be drawn in given angles to the tangents, and AC joining the points of contact, the rectangle under EL, EM and the square of EK will be to one another in a constant or given ratio. For take any other point e in the curve, and through E, e draw HR, hr parallel to AC , and let them meet the tangents in H, R and h, r ; and draw EQ, eq parallel to AR , and let them meet AC in Q, q . Then by similar triangles, as above, $ER \times EH : er \times eb :: EL \times EM : el \times em$; and by Cor. 5. Prop. V. $ER \times EH : er \times eb :: EQ^2 : eq^2$. But on account of the parallel lines, the triangles EKQ, ekq are similar, and $EQ^2 : eq^2 :: EK^2 : ek^2$. Consequently (II. v.) $EL \times EM : el \times em :: EK^2 : ek^2$.

Fig. 127.

PROP. VII.

The curve of a conic section cannot meet the curve of another conic section, or the curves of opposite hyperbolas, in more than four points.

For, if it be possible, let the curve of a conic section meet the curve of another conic section, or the curves of opposite hyperbolas, in the points A, B, D, C, E ; and

Fig. 137.

N

and

BOOK IV. and draw AB, BD, DC, CA . Let the straight lines EN, EH be drawn parallel to AB, AC , and let them meet BD, DC in N and H . Let the straight line BF be drawn, meeting the curves again in F, I , and the straight line EN in n . Let the straight lines FC, IC be drawn, and let them meet the straight line EH in K and L . Then, by Cor. 2. Prop. V. $HE:EN::LE:En$, and $HE:EN::KE:En$. Consequently (II. V.) $LE:En::KE:En$, and (14. V.) LE, KE are equal: which is absurd. The curve of the conic section, therefore, does not meet the curve of the other, or the curve of opposite hyperbolas, in five points.

Fig. 138. Cor. 1. If two conic sections touch one another, they will not meet each other in three other points. For, if it be possible, let the two sections have the common tangent in the point B , and meet one another in A, E, C , and of these let E be the intermediate point. Let BA, BC, CA be drawn; and let ET, EH be drawn parallel to BA, AC , and let them meet BT, BC in T and H . Through the point of contact B let BD be drawn, meeting the curves in D and d , and the straight line ET in N ; and let DC, dc be drawn, meeting EH in I and L . Then, by Cor. 4. Prop. V. HE is to ET as LE to EN , and IE is to EN in the same proportion, and therefore (9. V.) IE, LE are equal: which is absurd. The two points D, d therefore coincide, and the two sections meet in the five points A, B, D, C, E ; which by this Prop. is impossible.

Fig. 139. Cor. 2. If two conic sections AFD, AfD touch one another in the points A, D , they will not meet one another in any other point. For, if it be possible, let them meet one another in the point I , and let the straight line IF be drawn, meeting the tangents AB, DC in E, C , and the curve of the section AFD in F .

As

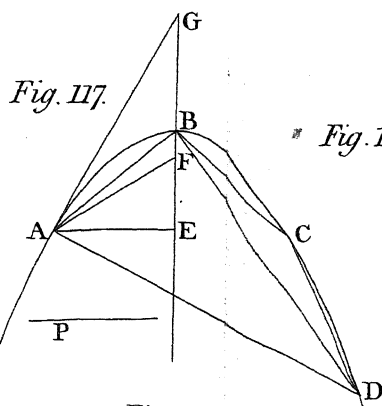
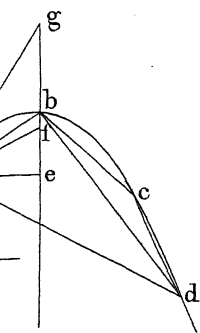


Fig. 117.

* Fig. 123.

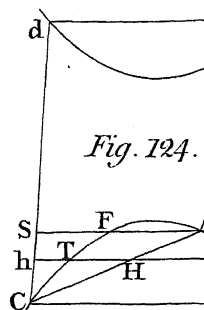
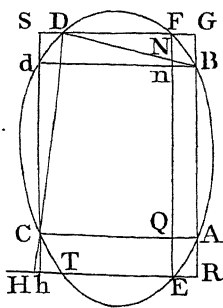


Fig. 124.

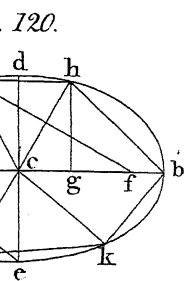


Fig. 119.

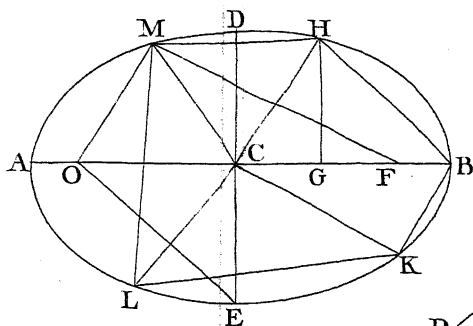


Fig. 125.

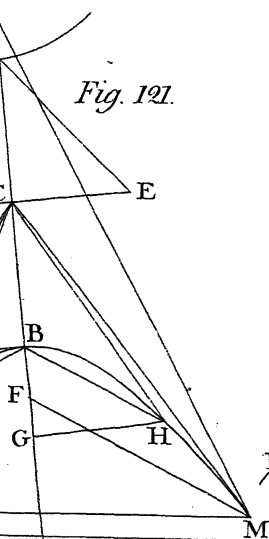
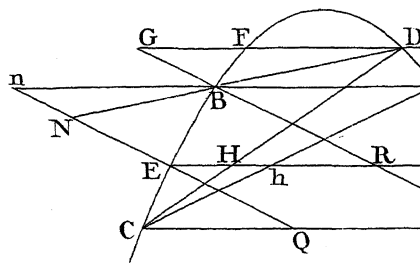


Fig. 121.

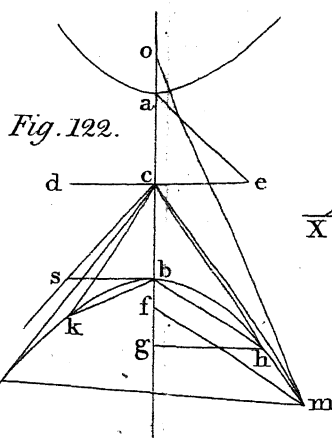


Fig. 122.

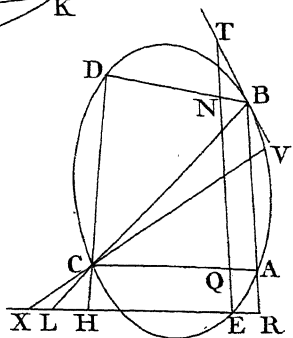


Fig. 126.

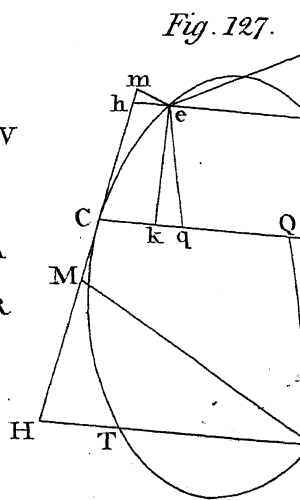
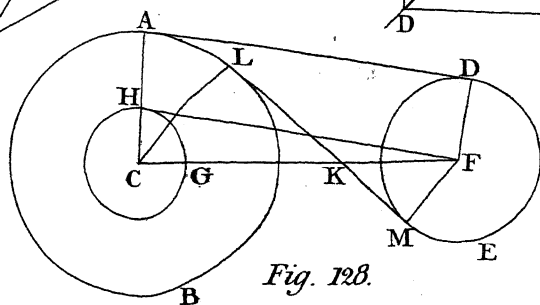
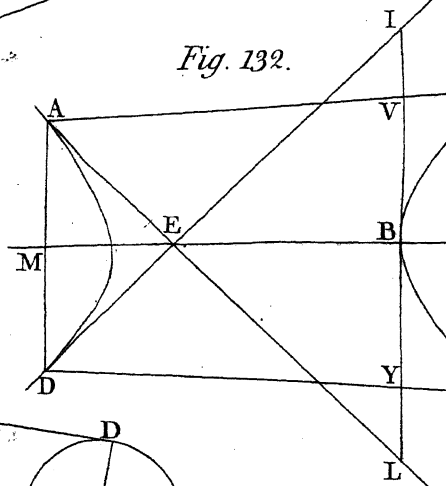
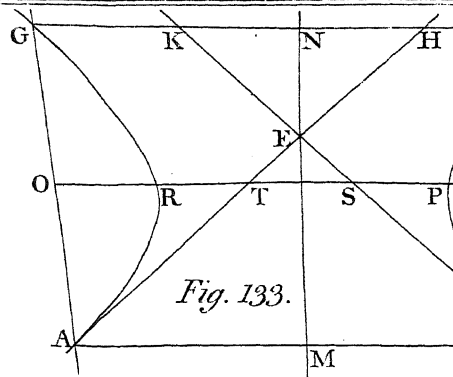
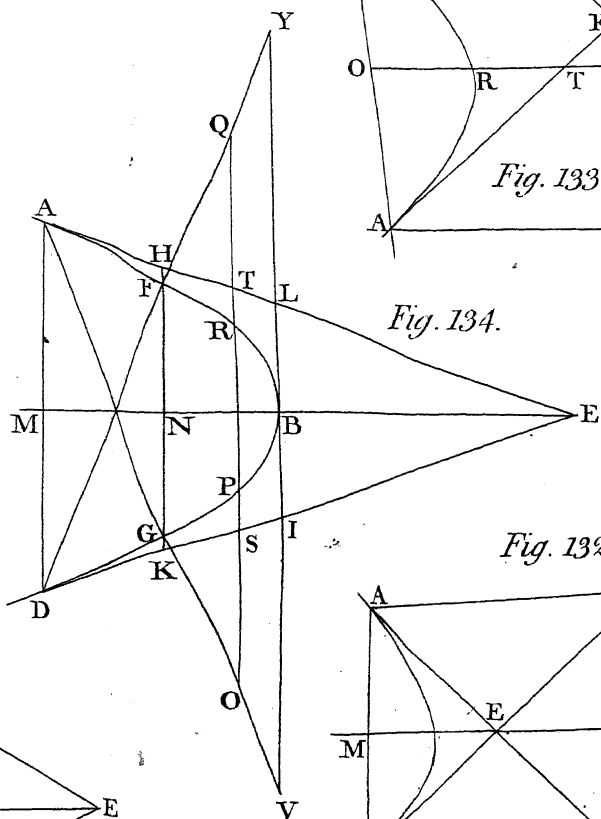
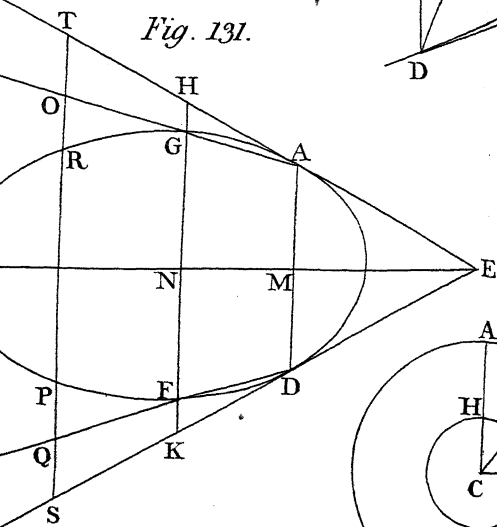
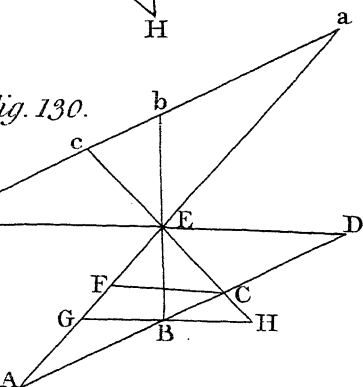
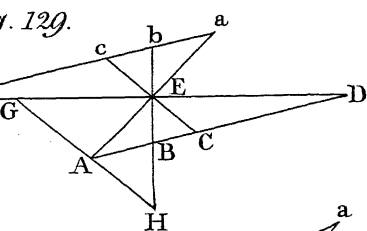
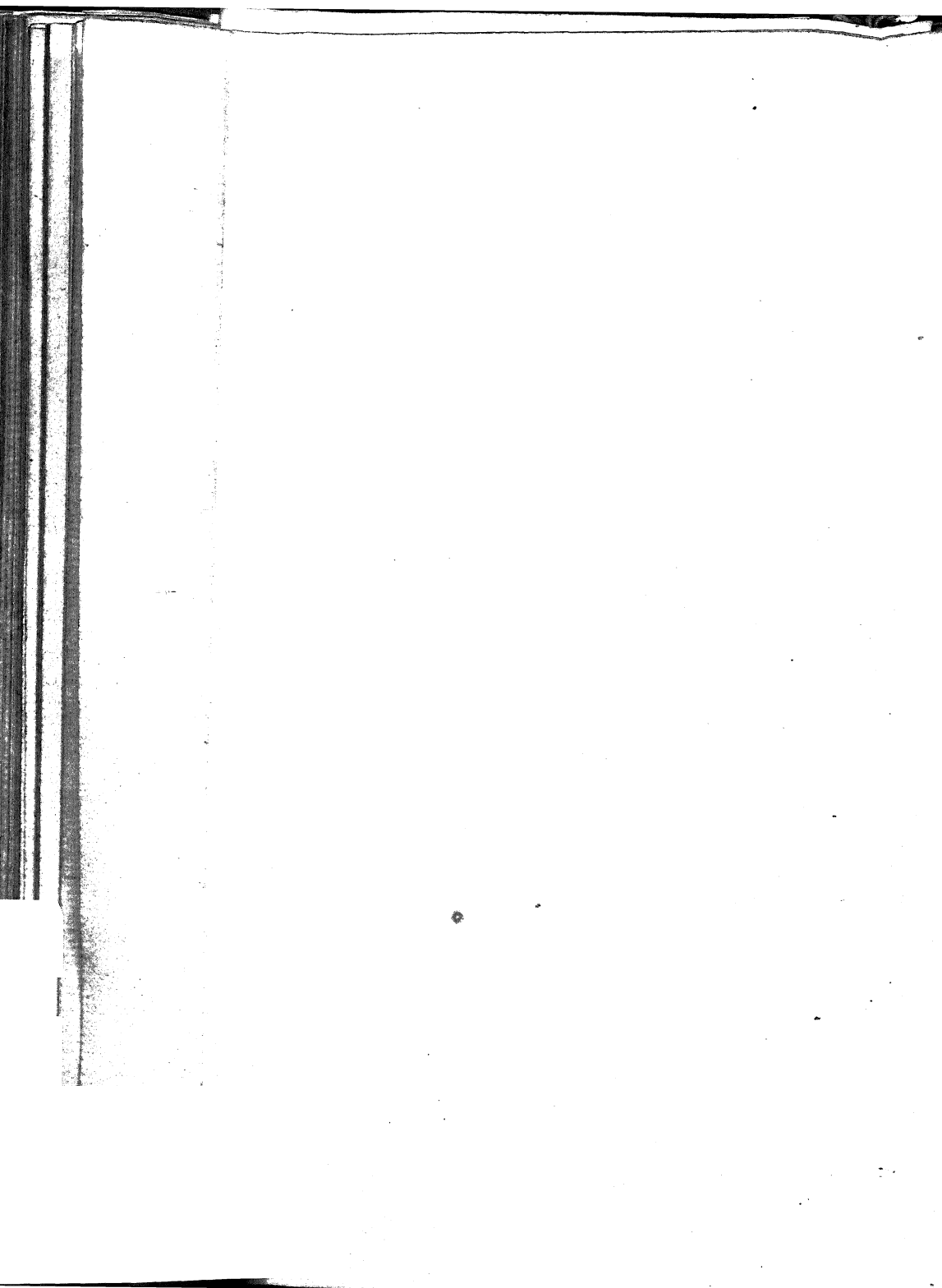


Fig. 127.





As the sections touch one another in D , by the preceding Cor. they do not meet one another in three other points. Let the straight line IG therefore meet the curve of the section AfD in f . Then, by Cor. 2. Prop. XVII. Book I. $BE^2 : GE^2 :: IB \times BF : FG \times GI$; and $BE^2 : GE^2 :: IB \times Bf : fG \times GI$. Consequently (II. v.) $IB \times BF : FG \times GI :: IB \times Bf : fG \times GI$, and by alternation $IB \times BF : IB \times Bf :: FG \times GI : fG \times GI$; and therefore (I. vi.) $BF : Bf :: FG : fG$. Hence (12. v.) $BF : Bf :: BG : BG$, and therefore BF, Bf are equal: which is absurd. Consequently the Cor. is evident.

BOOK
IV.

PROP. VIII.

If a straight line touch a conic section, and a straight line perpendicular to it be drawn through the point of contact, and meeting the axis or axes of the section, the segment of the perpendicular between the point of contact and the axis of a parabola, or between the point of contact and the transverse axis of the section, will be the least of all straight lines which can be drawn from the same point in the axis, and on the same side of it, to the curve; but the segment of the perpendicular between the point of contact and the conjugate axis of the ellipse will be the greatest of all straight lines which can be drawn from the same point in the axis, and on the same side of it, to the curve.

Let the straight line PR touch a conic section in the point P , and let the straight line PK , perpendicular to the tangent, meet AB the axis of a parabola, or the transverse axis of the section, in the point K , and let it meet DE the conjugate axis of the ellipse in M ; the segment PK is the least of all the straight lines which can be drawn from K , on the same side of AB , to the

Fig. 140.

141.

142.

BOOK
IV.

curve; and in the ellipse the segment MP is the greatest of all the straight lines which can be drawn from M , on the same side of DE , to the curve.

Let the tangent PR meet the axis AB in R ; and draw FG a double ordinate to AB , and let it meet AB in F , and the curve again in G , and draw RG . Then, as the angles at F are right angles, and as PF is equal to FG , we have (4. i.) PR equal to GR , GR equal to PK , and KGR a right angle, being equal (8. i.) to the angle KPR . Let the straight line VH , touching the section in the vertex V , meet the tangent PR in H , and draw KH . Then in the parabola the square of

Fig. 140. VH is to the square of PH , as the parameter of the axis AB to the parameter of the diameter passing through P , by Prop. IV. Book III. and therefore, by Cor. 1. Prop. XI. Book III. the square of VH is less than the square of PH . Consequently, as KH is common to the two right angled triangles KVH , KPH , the square of KV (47. i.) must be greater than the square of KP , and KP must be less than KV . Again,

Fig. 141. in the ellipse or hyperbola, c being the center, by
142. Prop. V. Book II. the square of VH is to the square of PH as the square of cd to the square of the semidiameter parallel to PH ; and therefore, by Prop. XI. Book II. the square of VH is less than the square of PH . Consequently, for the same reasons as in the pa-

Fig. 140. parabola, KP is less than KV . If therefore with K as a
141. center, and KP as a distance, in each section, a circle
142. be described, its circumference will pass through G , and cut AB within the section*; and as KPR , KGR are right angles, PR and GR (16. iii.) are tangents to

* The circle is intentionally omitted in the figures. The description of it would have made them more complex, and not rendered the Proposition or either of the Corollaries more perspicuous.

VII. the circumference of the circle cannot meet the curve of the section in any other point besides P and G ; and therefore KP is the least of all straight lines which can be drawn from K , on the same side of AB , to the curve.

In the ellipse draw PN a double ordinate to the conjugate axis DE , and let it meet DE in L , and the curve again in N . Let the tangent PR meet DE in T , and draw NT , NM . Then it may be proved, as above, that MN is equal to MP , NT equal to PT , and that the angle MNT is a right angle, being equal to the angle MPT . In the ellipse let the straight line dv , touching the section in the vertex D , meet the tangent PR in v , and draw Mv . Then, by Cor. 3. Prop. III. Book II. dv is parallel to the axis AB , and by Prop. V. Book II. the square of dv is to the square of rv , as the square of cb to the square of the semidiameter parallel to rv ; and therefore, by Prop. XI. Book II. the square of dv is greater than the square of rv . Consequently, as the angles at D and P are right angles, and Mv common to the two triangles MDv , MPv , the square of MP (47. i.) must be greater than the square of MD . If therefore with M as a center, and MP as a distance, a circle be described, its circumference will pass through N , it will cut DE without the ellipse, and, for the same reasons as above, the straight lines TP , TN will touch it and the section in the points P , N . Consequently, by Cor. 2. Prop. VII. the circumference of the circle cannot meet the curve of the section in any other point besides P and N ; and therefore MP is the greatest of all straight lines which can be drawn from M , on the same side of DE , to the curve of the ellipse.

Fig. 141.

Cor. 1. If a straight line as PG be a double ordinate

BOOK
IV.

Fig. 140.
141.
142.

to AB the axis of a parabola, or the transverse axis of a conic section, a circle touching the section in P , and passing through G , will also touch the section in G ; and the other parts of its circumference will fall wholly within the section. For PG will be in the circle, and, being bisected by AB at right angles, the center of the circle (Cor. 1. iii.) will be in AB . Let K be the center, and draw KP , and let PR be common to the circle and section, according to the third Definition. Then (18. iii.) KPR is a right angle; and KG , GR being drawn, it may be proved, as above, that the angles KGR , KPR are equal. Consequently GR will touch the circle, (16. iii.) and it is evident, from Cor. 2. Prop. VI. Book III. that it also touches the section: Hence the Cor. is manifest.

Fig. 141.
142.

Cor. 2. If a straight line as PN be a double ordinate to DE the conjugate axis of an ellipse, or of opposite hyperbolas, a circle touching the ellipse or hyperbola BP in P and passing through N will also touch the ellipse or the opposite hyperbola in N . For let the axis DE meet the common tangent PR in T , and the ordinate PN in L . Then, as PN will be in the circle, the center of the circle (Cor. 1. iii.) will be in DE . Let M be the center, and draw MP , MN , NT . Then (4. i.) MP is equal to MN , and TP equal to TN ; and therefore (8. i.) the angle MNT is equal to the angle MPT , which is a right one. Hence (16. iii.) the circle touches the ellipse or the opposite hyperbola in N , and NT is the common tangent to the circle and section.

In this case it is evident, that the circle described with the center M , and the distance MP , falls without the ellipse, and without each of the opposite hyperbolas. For, by this Prop. MP in the ellipse is the greatest straight line which can be drawn from M to the curve; and in the hyperbolas, as TP , TN are the common tangents, it is evident that MP , MN are the least
straight

PROP. IX.

If from the vertex of the axis of a parabola, or from a vertex of the transverse axis of an ellipse or hyperbola, a segment be taken in the axis equal to its parameter, a circle described about this segment as a diameter will fall wholly within the section; but if from a vertex of the conjugate axis of an ellipse a segment be taken in the axis equal to its parameter, a circle described about this segment as a diameter will fall wholly without the section.

Let AB be an axis of a conic section, and in the hyperbola the transverse axis, and from the vertex A let the segment AC be taken in the axis equal to its parameter; the circle AEC described about AC as a diameter will fall wholly within the section, unless AB be the conjugate axis of the ellipse, and if AB be the conjugate axis of the ellipse, the circle will fall wholly without the section.

Fig. 143.
144.
145.
146.

For through A draw the straight line AD equal to AC , and at right angles to AB , and draw CD . Through any point F in AC draw FG an ordinate to AB , and let it meet the circumference of the circle in E , the curve of the section in G , and the straight line CD in K . In the ellipse and hyperbola draw from the vertex B the straight line BD , and let it meet FG in H ; but in the parabola draw DH parallel to the axis AB , and let it meet FG in H . Then, by Prop. II. Book III. AD , FK are parallel, and therefore (4. vi.) $AD : AC :: FK : FC$; and as AD is equal to AC , FK is equal to FC . Consequently $AF \times FK$ is equal to $AF \times FC$, and therefore (35. iii.) $AF \times FK$ is equal to the square of

$N 4$

EF .



BOOK E F. But, by Cor. 1. Prop. VI. Book II. and Prop. III.
 IV. Book III. the square of FG is equal to $AF \times FH$. In the parabola and hyperbola therefore, and when AB is the transverse axis of the ellipse, the square of FG is greater than the square of FE , and consequently the point G is without the circle. But if AB be the conjugate axis of the ellipse, as in Fig. 146. the square of FE will be greater than the square of FG , and therefore the point G will be within the circle.

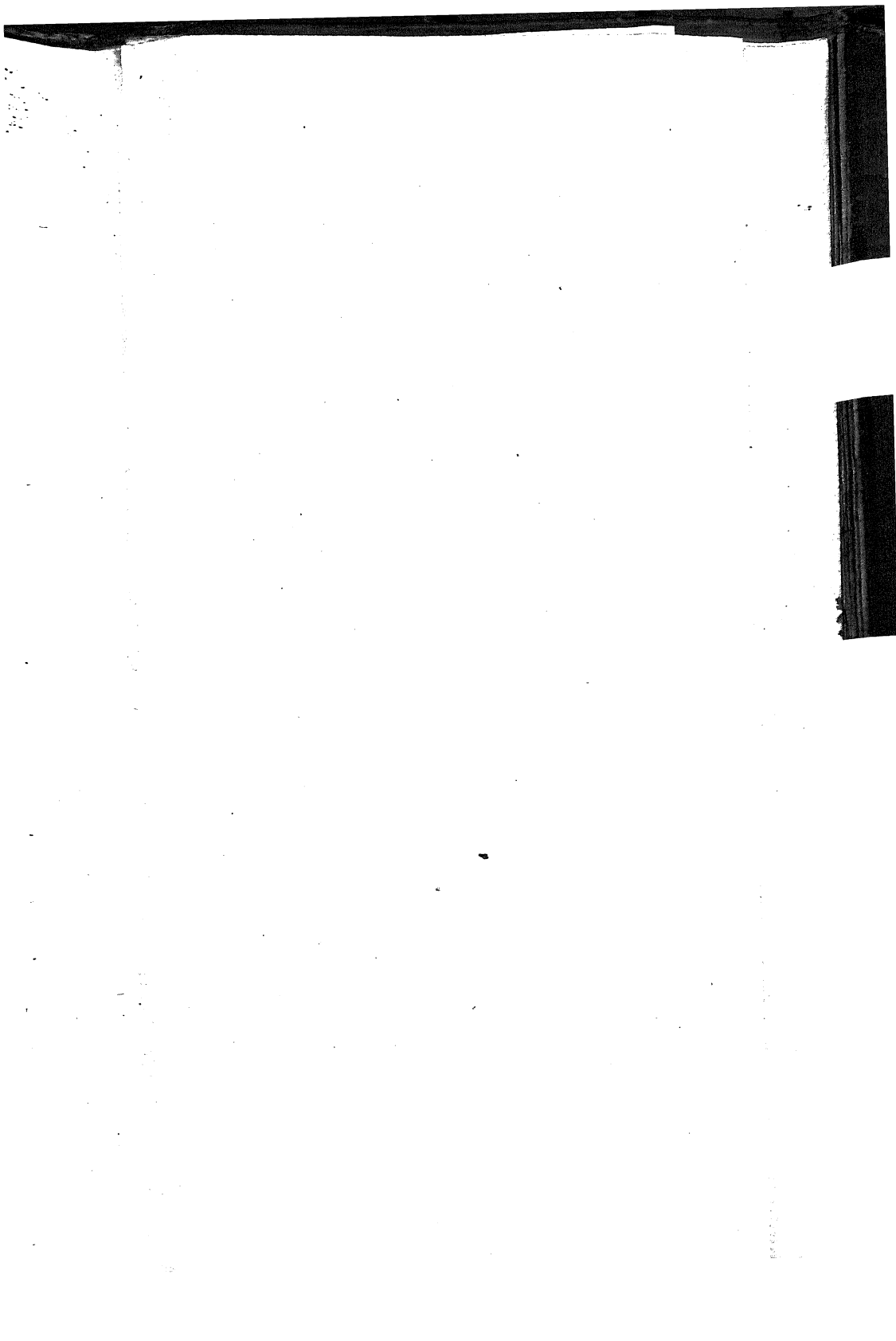
PROP. X.

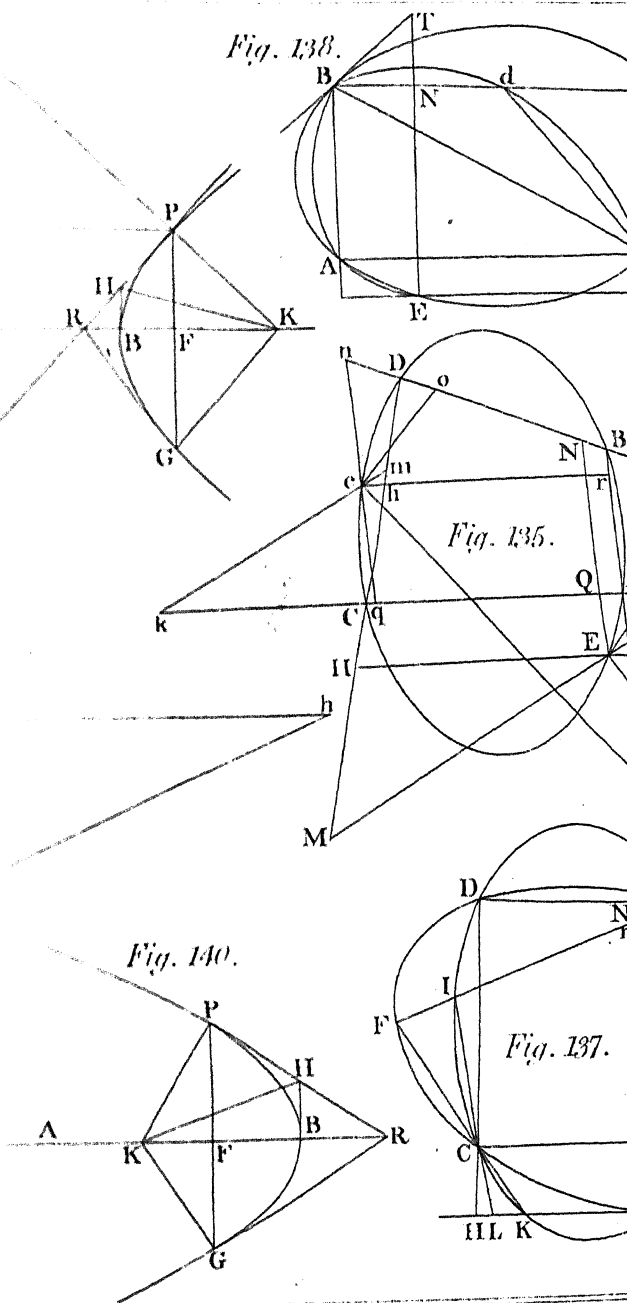
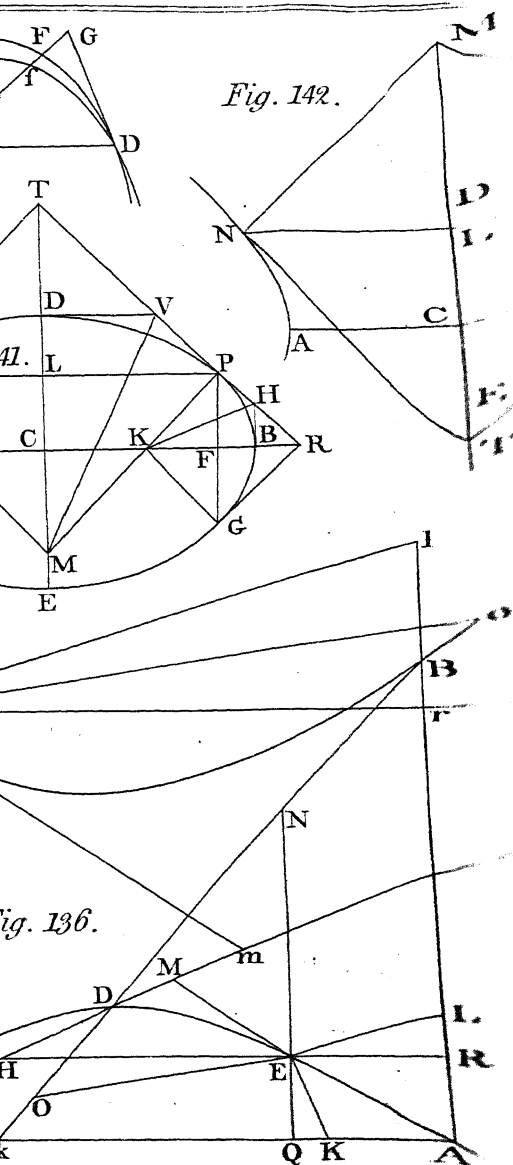
If from the vertex of the axis of a parabola, or from a vertex of the transverse axis of an ellipse or hyperbola, a segment be taken in the axis greater than its parameter, a circle described about this segment as a diameter will fall without the section on each side of the vertex; but if from a vertex of the conjugate axis of an ellipse a segment be taken in the axis less than its parameter, a circle described about this segment as a diameter will fall within the ellipse on each side of the vertex.

Fig. 147.
 148.
 149.

First, let AB be the axis of a parabola, or the transverse axis of an ellipse or hyperbola, and from the vertex A let the segment AC be taken in the axis greater than its parameter; the circle AIC described about AC as a diameter will fall without the section on each side of the vertex A .

In the ellipse let the point C be between the vertices A, B , and in each section let the straight line AD be drawn perpendicular to the axis AB , and equal to its parameter. In AD let the segment AF be taken equal to AC , and draw CF . In the ellipse and hyperbola let the straight line BD be drawn, and let it meet CF in E ; but in the parabola let the straight line DE be drawn parallel to the axis, and let it meet CF in E . In each





18.

B

each section let any point G be taken in CF , between the points E, F ; and draw GL parallel to the ordinates of AB , and let it meet DE in H , the circumference of the circle in I , the curve of the section in K , and the axis AB in L . Then, by Prop. III. Book III. and Cor. I. Prop. VI. Book II. the square of the ordinate LK is equal to the rectangle under AL, LH ; and on account of the equals AF, AC , the rectangle under AL, LG is equal to the rectangle under AL, LC , and therefore (35. iii.) equal to the square of IL . But the rectangle under AL, LG is greater than the rectangle under AL, LH , and therefore the square of IL is greater than the square of LK . Consequently the point I is without the section; and if the straight line GL meet the circumference of the circle again in M , the point M , for the same reasons as above, will be without the section. The circle AIC therefore falls without the section on each side of the vertex A .

BOOK
IV.

Secondly, let AB be the conjugate axis of an ellipse, and from the vertex A let the segment AC be taken in the axis less than its parameter; the circle AIC described about AC as a diameter will fall within the ellipse on each side of the vertex A . Fig. 150.

For let AC be greater than the axis AB . Let the straight line AD be drawn perpendicular to AB , and equal to its parameter. In AD let the segment AF be taken equal to AC , and draw BD, CF , and let them meet one another in the point E . In CF take any point G between the points F, E , and draw GL parallel to the ordinates of AB , and let it meet BD in H , the curve of the ellipse in K , the axis in L , and the circumference of the circle in I, M . Then, as above, it may be demonstrated, that the points I, M are within the ellipse; and therefore that the circle AIC falls within the ellipse on each side of the vertex A .

Cor.

BOOK
IV.

Cor. 1. If from the vertex of the axis of a parabola, or from a vertex of the transverse axis of an hyperbola, or from a vertex of either axis in an ellipse, a segment be taken in the axis equal to its parameter, a circle described about this segment as a diameter will have the same curvature with the section in the vertex. For, by the preceding Prop. in the parabola, and when AB is a transverse axis, the circle AEC falls wholly within the section, the diameter AC being equal to the parameter of the axis AB ; and it is evident, that if a segment be taken from A in AB less than AC , a circle described about it as a diameter will fall within the circle AEC . Again, by this Prop. if from A a segment be taken in AB greater than the parameter of AB , or greater than AC , in Fig. 147. it will fall without the section. In Fig. 143. therefore the circle AEC has the same curvature with the section in A . Also when AB is the conjugate axis of the ellipse, by the preceding Prop. the circle AEC falls wholly without the ellipse, AC being equal to the parameter of AB ; and it is evident, that if a segment be taken from A in AB greater than AC , the circle described about it as a diameter will fall without the circle AEC . Again, by this Prop. if from A a segment be taken in AB less than the parameter of AB , or less than AC , in Fig. 150. a circle described about it as a diameter will fall within the section. In this case therefore the circle AEC in Fig. 146. has the same curvature with the section in A , according to the fourth Definition.

Cor. 2. If a circle touch a conic section in the vertex of an axis, and have the same curvature with the section in the vertex, it will cut off from the axis a segment equal to its parameter. This is evident from the preceding Corollary.

PROP.

PROP. XI.

BOOK
IV.

If from a point in the curve of a conic section a double ordinate be drawn to the axis of a parabola, or to the transverse axis of the section, and if through the point in which it meets the curve again a diameter be drawn, and from the first mentioned point a double ordinate be drawn to this diameter; a circle touching the section in the first mentioned point, and passing through the other extremity of the last mentioned double ordinate, will not meet the section in any other point besides these two, and it will have the same curvature with the section in the point of contact.

From the point B in the curve of the conic section Fig. 153.
BAL let the straight line BA be drawn a double ordi- 154.
nate to DH the axis of the parabola BAL, or to DH 155.
the transverse axis of the section, and from A, the point 156.
in which it meets the curve again, draw the diameter
AE, and from B draw the double ordinate BF to AE;
the circle BKF, touching the section in B and passing
through F, the point in which the double ordinate BF
meets the curve again, will not meet the section in any
other point besides B, F, and it will have the same
curvature with the section in the point B.

For if the circle KFB pass through the point A, it
will touch the section also in A, by Cor. I. Prop. VIII.
and the other parts of the circumference will fall wholly
within the section; and if the section be an ellipse,
and BK be drawn, it will be an ordinate to the conju-
gate axis; and therefore if the circle KFB pass
through E, it will also touch the section in E, by Cor.
2. Prop. VIII. and the other parts of the circumfe-
rence will fall wholly without the ellipse. If there-
fore the circle KFB pass through A in any section, or
through

BOOK
IV.

through E in the ellipse, it will touch the section in the points B , A or B , E , and, by Cor. 2. Prop. VII. it will not pass through F , contrary to hypothesis. Hence it is evident that the circle KBF in any section is greater than a circle which touches the section in B , A , but in the ellipse it must be less than a circle which touches the ellipse in B , E ; and therefore, in any section, the circumference of the circle KBF meets the straight line BA without, but the straight line BE within the section. Let the circumference meet BA in K , and AE in α . If it be possible, let the circumference of the circle meet the section in L . Draw the common tangent BD , meeting the axis DH in D , and draw AD . Then, by Cor. 2. Prop. VI. Book III. AD will touch the section, and therefore, by Prop. II. Book III. AD , BF are parallel, and on account of the axis DH , the tangents (4. i.) AD , BD are equal. Let LM be drawn parallel to AD or FE , meeting the curve again in I , and the tangent BD in M . Then, by Prop. XIII. Book I. $BD^2 : AD^2 :: BM^2 : LM \times MI$; and therefore BM^2 is equal to $LM \times MI$; and as, by hypothesis, the point L is in the circumference of the circle, the point I (36. iii.) is also in the circumference of the circle. The circle KFB therefore, touching the section in B , meets the section in F , L , I , which, by Cor. 1. Prop. VII. is impossible. The circle KFB therefore does not meet the section in any point besides B , F ; and as the point K is without, and the point α within the section, the arch BKF will be without, and the arch $F\alpha B$ will be within the section.

The circle KFB will also have the same curvature with the section in the point B . For let any other circle as $BR L$, touching the section in B , be described; and, first, let $BR L$ be less than KFB . Then, as the straight

iii.) will be in the same straight line, and therefore the lesser circle BRL will fall wholly within the circle KFB . The circle BRL therefore cannot pass between the arch BQF and the curve of the section; and if it pass through A , or be less than a circle passing through A , it will fall wholly within the section, by Cor. 1. Prop. VIII. and therefore it cannot pass between the arch FKB and the curve of the section. But let the circle BRL be greater than the circle passing through A , and meet the straight line BA in v , and the curve of the section in L^* . Let the straight line LM be drawn parallel to AD , meeting the curve of the section again in I , and the tangent BD in M . Then, by Prop. XIII. Book I. $BD^2 : AD^2 :: BM^2 : LM \times MI$, and, on account of the equals BD , AD , the square of BM is equal to $LM \times MI$. The point I therefore (36. iii.) is in the circumference of the circle, and consequently, by Cor. 1. Prop. VII. it meets the section only in the points B , L , I . Again, it is evident that the circle BRL meets BF within, and BA without the section, and therefore that the curve of the section between the points F , L are without the circle. In the tangent BM therefore take any point T between B and M , and let the straight line TP be drawn parallel to AD , meeting the section in P , s , and the circle BRL in R , p . Then, as above, it may be demonstrated that BT^2 is equal to $PT \times Ts$. But the square of BT (36. iii.) is equal to the rectangle $RT \times Tp$, and therefore the rectangles $PT \times Ts$, $RT \times Tp$ are equal. Consequently $PT : TR :: Tp : Ts$, and as PT is greater than TR , Tp is greater than Ts . The arch BpI therefore falls within the section.

* Such lines as could easily be supplied by the mind of the reader are omitted in Fig. 155. and 156.



BOOK
IV.

Fig. 155.

Secondly, let the circle BRL be greater than the circle KFB ; and it may then be demonstrated, as above, that the circle BRL falls without KFB , and therefore that it cannot pass between the arch FKB and the section. Again, if the circle BRL pass through E in the ellipse, or be greater than the circle passing through E , it will fall wholly without the ellipse, and therefore it will not pass between the ellipse and the arch FQB . But in any section let the circle BRL meet AE within the section, and the curve of the section in L . Draw the straight line LM parallel to AD , and let it meet the section in I , and the tangent BD in M . In BD take any point T between B and M . Draw TR , and let it meet the section in P and S , and the circle BRL in R and p . Then, as above, it may be demonstrated that the arch PI is without the section. The circle KFB therefore has the same curvature with the section in the point of contact B , according to Def. IV.

Cor. From the above, and Cor. 1. Prop. X. it is evident that only one circle, touching a conic section in a given point, can have the same curvature with the section in that point.

PROP. XII.

If a circle touch a conic section, and have the same curvature with the section in the point of contact, it will cut off from the diameter of the section passing through the point of contact a segment equal to its parameter.

If the point of contact be the vertex of an axis, the Proposition has been demonstrated, as stated in Cor. 2.

Fig. 160.
161.
162.

Prop. X. but if the circle KBF touch the section in the point B , which is not a vertex of an axis, let every thing remain as in the preceding Prop. and let KBF be the circle having the same curvature with the section in the

the point B. Let the circumference of the circle KBF therefore meet BC , drawn through B the point of contact and c the center, in the point N , if the section be an ellipse or hyperbola; or let it meet the diameter BN in the point N , if the section be a parabola; in either case the segment BN is equal to the parameter of the diameter BCN .

First, let the section be an ellipse or hyperbola, and draw the diameter cv parallel to the tangent DA , and the diameter TM parallel to the tangent BD , and meeting BF in G . Let the tangent DA meet the diameter TM in M , and the diameter BC in L . Let the diameter AE meet its ordinate BF in I . Then, on account of the axis DC and its ordinate AB , the semidiameters CA , CB are equal; and by similar triangles $LA : AC :: BI : CI$; and $AC : AM :: CI : IG$. Consequently,

$$LA : AC : AM$$

$$BI : CI : IG,$$

and (22. v.) $LA : AM :: BI : IG$; and therefore (22. vi.) $LA \times AM : BI \times IG :: LA^2 : BI^2$, or, by the above, as AC^2 to CI^2 . But, by Cor. 2. Prop. IV. Book II. the diameters TM , BL are conjugate, and therefore $LA \times AM$ is equal to cv^2 , by Cor. 2. Prop. IX. Book II. Consequently $cv^2 : BI \times IG :: AC^2 : CI^2$; and, by alternation, $cv^2 : AC^2 :: BI \times IG : CI^2$; and, by Prop. V. Book II. $cv^2 : AC^2 :: BI^2 : AI \times IE$. Consequently, by the tenth Lemma, (and 12. v. and 3. ii.) $cv^2 : AC^2 :: GB \times BI : AC^2$, and therefore (14. v.) cv^2 is equal to $GB \times BI$. But, by Prop. V. Book II. $AD^2 : BD^2 :: cv^2 : CT^2$; and therefore, on account of the equals AD , BD , the square of cv is equal to the square of CT . The square of CT is therefore equal to $GB \times BI$; and $NB \times BC$ is equal to $FB \times BG$, by the seventh Lemma. If therefore NB be bisected in P , the rectangle under PB , BC will

Fig. 160.
161.

BOOK will be equal to $GB \times BI$, or to CT^2 . But CT^2 is equal to the rectangle under CB and half the parameter of CB . Consequently the segment BN is equal to the parameter of the diameter CB .

Fig. 162. Secondly, let the section be a parabola, and let the diameter AE meet its ordinate BF in E . Draw to the diameter BN the ordinate AC , meeting the diameter BN in C , and BF in H . Then, on account of the equals AD, BD , the parameter of the diameters AE, BC will be equal, by Prop. IV. Book III. and by Cor. Prop. V. Book III. AE, BC are equal, and as AE, BC are parallel, the triangles AEH, BCH are equiangular. Consequently (4. vi.) $AE : EH :: BC : BH$, and therefore (14. v.) EH, HB are equal. Consequently $FB : BE :: BE : BH$, and therefore $FB \times BH$ is equal to BE^2 . But, by the seventh Lemma, $FB \times BH$ is equal to $NB \times BC$; and as BE, AC are equal, BE^2 is equal to AC^2 . The rectangle $NB \times C$ therefore is equal to the square of AC . Moreover the square of AC is equal to the rectangle under BC , and the parameter of the diameter BN ; and therefore the rectangle $NB \times C$ is equal to the rectangle under the absciss BC , and the parameter of the diameter BN . The segment BN is therefore equal to the parameter of the diameter drawn through B the point of contact.

Cor. 1. If from the vertex of a diameter of a parabola, or from a vertex of a transverse diameter of an hyperbola, or from a vertex of any diameter of an ellipse, a segment be taken in the diameter equal to its parameter, a circle touching the section in the vertex, and passing through the other extremity of the segment, will have the same curvature with the section in the vertex. This is evident from Cor. 1. Prop. X. and the above.

Cor. 2. If through O , the focus of the parabola, the straight

meter of the diameter BC .

For through O draw OX parallel to BD , and let it meet BC in X . Then, by Cor. Prop. XII. Book III. BX is equal to BO ; and by the seventh Lemma $RB \times BO$ is equal to $NB \times BX$. Consequently BR is equal to BN .

Cor. 3. The rest remaining as above in the ellipse and hyperbola, if the straight line BR drawn through the focus O meet the circle again in R , and XS be the transverse axis, then BR will be to the diameter CT as CT is to XS . For let the diameter CT meet BR in Y , and then, by the seventh Lemma, the rectangle RBV is equal to the rectangle FBG . But by the above the rectangle FBG is equal to twice the square of CT , and therefore the rectangle RBV is equal to twice the square of CT . Consequently $RB : \text{the whole diameter } CT :: \text{the semidiameter } CT : BV$. But, by Cor. Prop. XVI. Book II. BV is equal to CX , and therefore (15. v.) $BR : \text{the whole diameter } CT :: \text{the whole diameter } CT : XS$.

Fig. 160.
161.

PROP. XIII.

If a circle touching an ellipse or hyperbola have the same curvature with the section in the point of contact, its semidiameter will be to the semidiameter of the section conjugate to that passing through the point of contact, as the square of the same semidiameter of the section to the rectangle under the semiaxes.

For if the circle touch the section in the vertex of an axis, then every thing remaining as in Prop. IX. by the Definition of a parameter, and inversion, AC is to the axis parallel to the common tangent at A , as the same axis to the axis AB . Consequently (15. v. and

O
I. iv.)

Fig. 144.
145.
146.



BOOK 1. iv.) as the semidiameter of the circle AEC to the
 IV. femiaxis parallel to the common tangent at A , so is the square of the same femiaxis to the rectangle under the semiaxes.

Fig. 160. But if the circle of curvature do not touch the sec-
 161. tion in the vertex of an axis, let every thing remain as in Prop. XII. and let UB be the semidiameter of the circle; and then UB will be to CT as CT^2 to the rectangle under the semiaxes.

For let CX be the transverse semiaxis, and CH the conjugate semiaxis. From the center C draw the perpendicular CA to the tangent BD . Let UB meet the circumference again in W , and draw WN . Then (31. iii.) the angle WNB is equal to the angle CAB , and the angle NBW (29. i.) equal to the angle BCA . Consequently (4. vi.) $CB : CA :: WB : BN$, or (15. v.) as UB to PB ; and therefore $CA \times UB$ is equal to $CB \times PB$. But, by Prop. XII. and the Definition of a parameter, $CB \times PB$ is equal to CT^2 , and therefore $UB : CT :: CT : CA$. Consequently (1. vi.) $UB : CT :: CT^2 : CT \times CA$. But, by Cor. 1. Prop. XIX. Book II. $CT \times CA$ is equal to $CX \times CH$; and therefore $UB : CT :: CT^2 : CX \times CH$.

Cor. By the above the square of CT is equal to the rectangle under UB , CA .

PROP. XIV.

If from the center of a circle, touching a conic section, and having the same curvature with the section in the point of contact, a perpendicular be let fall upon a straight line drawn from the point of contact through the nearest focus, a straight line drawn from the point of concurrence to the point in which the diameter of the circle, passing through the point of contact, cuts the focal axis will be at right angles to this diameter of the circle.

From

From H the center of the circle LPM , touching the conic section AP in the point P and having the same curvature with the section in P , let the perpendicular HK be drawn to the straight line PK passing through F the nearest focus; the straight line KI , drawn from K the point of concurrence to I the point in which the diameter PH of the circle cuts the focal axis AI , is at right angles to HP .

BOOK
IV.

Fig. 157.
158.
159.

For first let the section be an ellipse or hyperbola of which C is the center, AC the transverse semi-axis, and DC the conjugate semi-axis. Let NP be the common tangent, and CT the semidiameter of the section parallel to NP ; and draw CG , FN perpendicular to NP . Then, by Cor. Prop. XIII. the square of CT is equal to the rectangle under HP , CG ; and, by Prop. XVII. Book II. the square of CD is equal to the rectangle under IP , CG . Consequently $CT^2 : CD^2 :: HP \times CG : IP \times CG$; and therefore (I. vi.) $CT^2 : CD^2 :: HP : IP$. But, by Prop. XIX. Book II. (and 22. vi.) $CT^2 : CD^2 :: FP^2 : FN^2$; and, on account of the similar triangles, $FP^2 : FN^2 :: HP^2 : PK^2$. Consequently (II. vi.) $HP^2 : PK^2 :: HP : IP$; and therefore (I. vi.) $HP^2 : PK^2 :: HP^2 : HP \times IP$, and (I4. v.) PK^2 is equal to $HP \times IP$. Hence $HP : PK :: PK : IP$, and (6. vi.) PIK is a right angle.

Fig. 157.
158.

Secondly, let the section be a parabola, and let the common tangent NP meet the axis AI in G . Then, by Cor. Prop. IX. Book III. GF is equal to FP , and therefore (6. i.) the angle FGP is equal to the angle FPG . But as each of the angles GPI , HKP is a right angle, the angles FPG , FPI together are equal to the angles KPI , PHK together. Consequently the angle IGP is equal to the angle PHK ; and therefore (4. vi.) $GI : IP :: PK : PH$, and $HP \times PI$ is equal to $GI \times PK$. But, by Cor. 2. Prop. XII. (and 3. ii.) PK is

Fig. 159.

BOOK equal to half the parameter of the diameter passing
IV. through P, and therefore, by Prop. XI. Book III. PK
is equal to GI. Consequently $HP \times PI$ is equal to
 PK^2 . As above, therefore, $HP : PK :: PK : IP$, and
(6. vi.) PIK is a right angle.

Fig. 157. Cor. If a straight line NG touch a conic section in P,
158. and PH at right angles to it meet the focal axis AB in
159. I, and if IK perpendicular to PH meet in the point K
the straight line PL drawn through the nearest focus
F, and, lastly, if KH at right angles to PL meet PH
in H, the point H will be the center of the circle which
touches the section, and has the same curvature with
it in P.

SCHOLIUM.

Fig. 160. If a body B revolve in a space void of resistance about
the center C in the curve BAF, and if the circle BKF
touch the curve in the point B, and have the same cur-
vature with it in that point; and if the straight line
BCN meet the circle again in N, and Ca be perpendi-
cular to the common tangent Ba, Sir Isaac Newton
has demonstrated, in Cor. 3. to Prop. VI. Lib. I. of the
Principia, that the centripetal force is reciprocally as
 $Ca^2 \times NB$, or directly as $\frac{1}{Ca^2 \times NB}$.

By means of this expression, and the properties of
osculating circles demonstrated in the preceding Pro-
positions, the centripal forces of bodies moving in co-
nic sections may be easily ascertained, as in the follow-
ing examples.

Fig. 160. 1. Let the body revolve in the ellipse xhs, and let
the law of centripetal force tending to c the center be
required.

Every thing remaining as in Prop. XIII. the centri-
petal force, according to the Newtonian expression, is
reciprocally as $Ca^2 \times BN$. But, by Cor. 1. Prop.
XIX.

XIX. Book II. $CT \times CQ = XC \times CH$, and therefore $CQ^2 = \frac{XC^2 \times CH^2}{CT^2}$; and by the Definition of a parame- BOOK
IV.

ter, and Prop. XIII. $BN = \frac{4CT^2}{2CB} = \frac{2CT^2}{CB}$. Consequent-

ly $CQ^2 \times BN = \frac{XC^2 \times CH^2}{CT^2} \times \frac{2CT^2}{CB} = \frac{XC^2 \times 2CH^2}{CB}$;

and therefore, as $XC^2 \times 2CH^2$ is a constant quantity, the centripetal force is reciprocally as $\frac{1}{CB}$, or directly as the distance CB .

2. Let a body P move in an ellipse or hyperbola PA , and let the centripetal force tending to the focus F of the section be required.

The rest remaining as in Prop. XIV. let the straight line PF meet the circle again in L , and, by Cor. 3. Fig. 157.
158.

Prop. XII. $LP : 2CT :: 2CT : 2CA$; and there-

fore $LP = \frac{4CT^2}{2CA} = \frac{2CT^2}{CA}$. Again, by Prop. XIX. Book

II. (and 22. vi.) $FN^2 : FP^2 :: CD^2 : CT^2$, and $FN^2 = \frac{FP^2 \times CD^2}{CT^2}$. But, the Newtonian general expression

being adapted to the present Figures, the centripetal force is reciprocally as $FN^2 \times LP$; and therefore this

force, by the above, is reciprocally as $\frac{FP^2 \times CD^2}{CT^2} \times$

$\frac{2CT^2}{CA} = \frac{FP^2 \times 2CD^2}{CA}$. Again, if the letter L be put for

the parameter of the transverse axis, $2AC : 2CD ::$

$2CD : \frac{4CD^2}{2AC} = \frac{2CD^2}{AC} = L$. Consequently as L , or its

value, is constant, the centripetal force is reciprocally as FP^2 , or inversely as the square of the distance.

3. Let a body P move in the curve of a parabola PA , and let the law of centripetal force tending to the focus F be required.

BOOK
IV.

Fig. 159.

Every thing remaining as in Prop. XIV. let the straight line PF meet the circle again in L , and let FN be drawn perpendicular to the tangent PG . Then, by Cor. 2. Prop. XII. of this, and Cor. 2. Prop. XI. Book III. $LP = 4FP$; and, by Prop. X. Book III. $FN^2 = FP \times AF$. Again by the Newtonian general expression, adapted to this Figure, the centripetal force is reciprocally as $FN^2 \times LP$. By the above therefore the centripetal force is reciprocally as $FP \times AF \times 4FP = FP^3 \times 4AF$. Consequently, as $4AF$ is constant, the centripetal force is reciprocally as FP^3 .

PROP. XV.

If three straight lines touch a conic section, or opposite hyperbolas, any one of them will be harmonically divided in its point of contact, the points in which it meets the other two, and the points in which it meets the straight line joining their points of contact.

Fig. 151.
152.
165.
166.
167.

Let the three straight lines QR , RP , PN touch a conic section EN , or opposite hyperbolas a , N in the points a , E , N ; any one of them as RP is harmonically divided in its point of contact E , the points R , P in which it meets the other tangents, and the point A in which it meets the line aN joining their points of contact.

Fig. 151.
152.

Case 1. First, let the tangents QR , NP be parallel; and then, by the Cor. to Prop. XIII. Book I. $RE : EP :: QR : NP$. But (4. vi.) $QR : NP :: RA : AP$, and therefore (II. v.) $RA : AP :: RE : EP$.

Fig. 165.
166.
167.

Case 2. Let the tangents QR , NP meet one another in the point B ; and through the point P draw PH parallel to QR , and let it meet the curve of the section in G , H , and aN in I . Then, by Prop. XIII. Book I. $RE^2 : EP^2 :: QR^2 : HP \times PG$. But, by Prop. XVII. Book I. $HP \times PG$ is equal to PI^2 . Consequently $RE^2 : EP^2$

$EP^2 :: QR^2 : PI^2$, and (22. vi.) $RE : EP :: QR : PI$. BOOK
 But (4. vi.) $QR : PI :: RA : AP$, and therefore (II. IV.
 v.) $RA : AP :: RE : EP$.

PROP. XVI.

If two straight lines touching a conic section, or opposite hyperbolas, meet one another, a secant passing through the point of concurrence will be harmonically divided in the point of concurrence, the points in which it meets the curve or curves, and the point in which it meets the straight line joining the points of contact.

Let the two straight lines EA , FA , touching the section EDF , or the opposite hyperbolas E , F , in the points E , F , meet one another in A , and let the straight line AB meet the curve or curves in B , D , and the straight line EF in C ; the straight line AB is harmonically divided in the points A , D , C , B .

Fig. 151.

152.

168.

169.

For through B , D draw the straight lines GM , HK parallel to EF , and let them meet the tangents EA , FA in G , M , and H , K and the curve or curves in B , L , and D , I . Then, by Prop. IV. GB , LM are equal, and therefore GL is equal to BM ; also HD , IK are equal, and therefore HI is equal to DK . By equiangular triangles GB is to HD as AB to AD , and BM , or its equal GL , is to DK , or its equal HI , in the same proportion. Consequently (II. v.) $GB : HD :: LG : IH$, and by alternation $GB : LG :: HD : IH$; and therefore (22. vi.) $BG \times GL : DH \times HI :: GB^2 : HD^2$. But, on account of the parallels, $GB^2 : HD^2 :: AB^2 : AD^2$; and, by Prop. XIII. Book I. $BG \times GL : DH \times HI :: EG^2 : EH^2$. Also, on account of the parallels, (10. vi.) $EG^2 : EH^2 :: CB^2 : CD^2$; and therefore (II. v.) $AB^2 : AD^2 :: CB^2 : CD^2$. Consequently (22. vi.) $AB : AD :: CB : CD$.

If AB bisect EF in C , by Cor. I. to Prop. V, Book

BOOK III. it will be a diameter. If therefore in this case A B
 IV. meet the curve of the section in two points, or the
 curve of each of the opposite hyperbolas in one, it must
 be a diameter of an ellipse, or a transverse diameter of
 an hyperbola; as a diameter of a parabola can meet
 the curve in one point only, and in the hyperbola a se-
 cond diameter does not meet either of the opposite
 curves. Consequently, by Cor. 3. Prop. VII. Book II.
 $AB : AD :: BC : CD$.

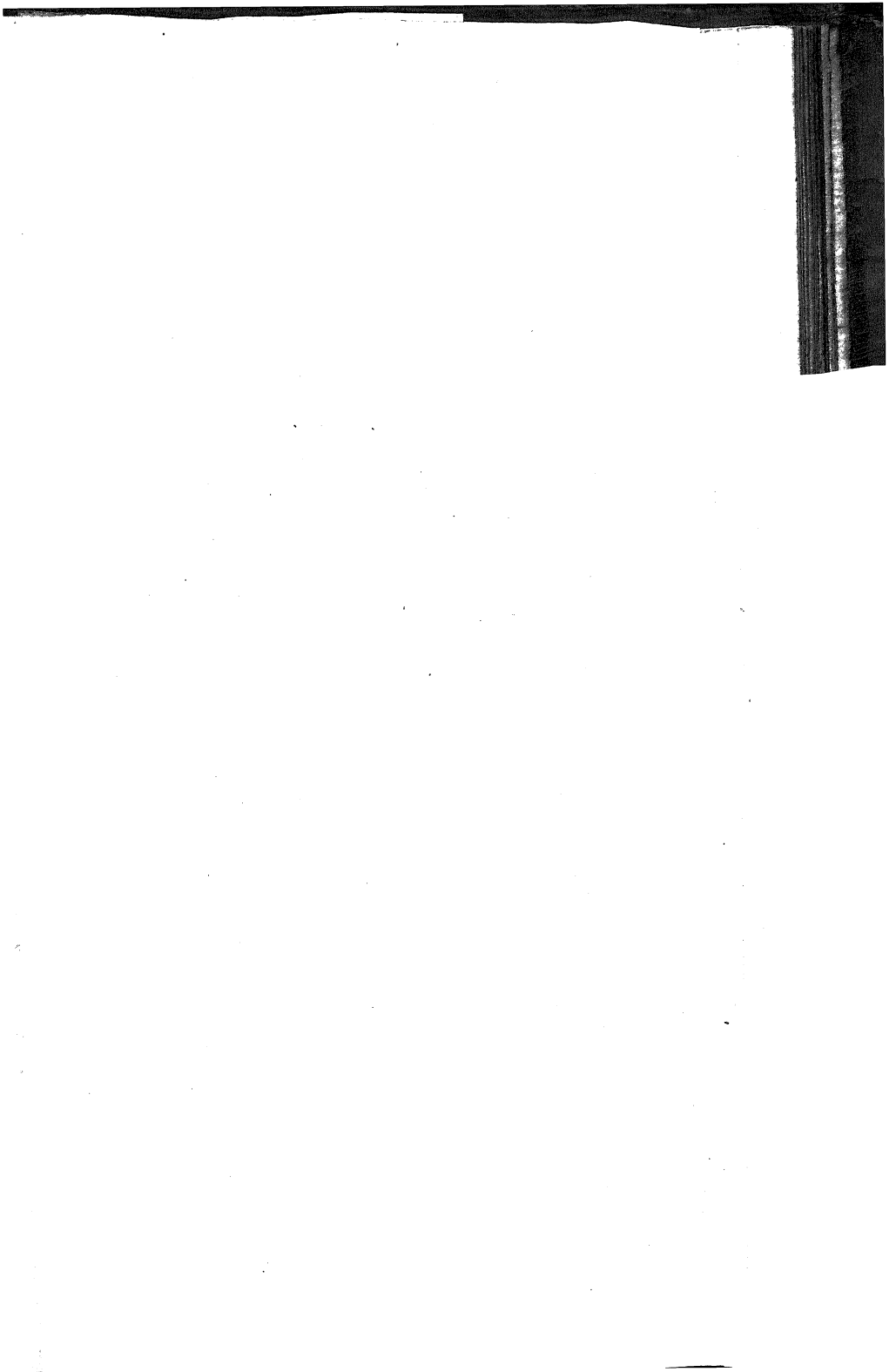
PROP. XVII.

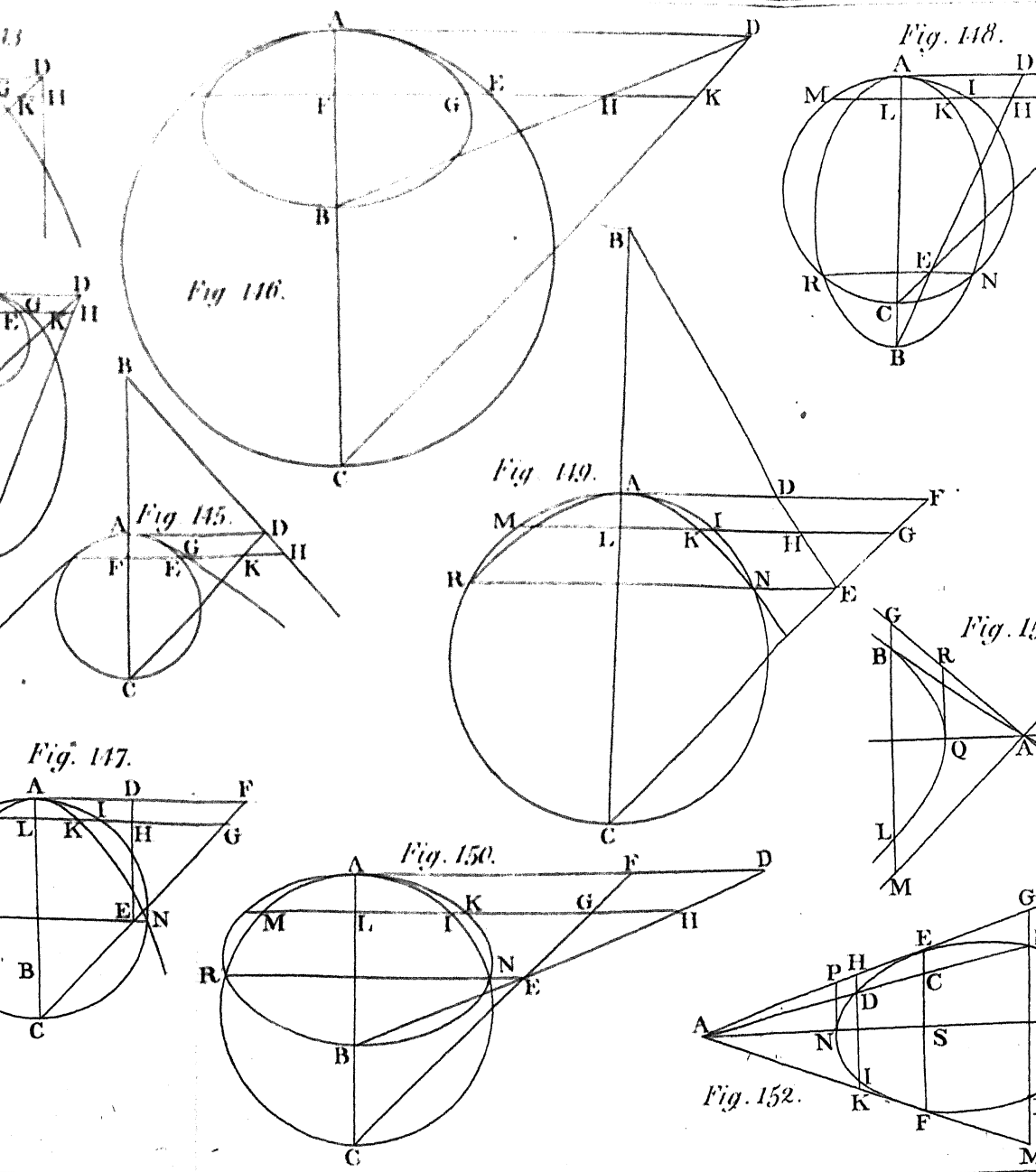
If two straight lines touching a conic section, or opposite hyperbolas, meet one another, and a secant pass through the point of concurrence, tangents passing through the points in which the secant meets the curve or curves will either be parallel, or they will meet one another in the straight line joining the points of contact of the two first mentioned tangents.

Fig. 170. Let the straight lines κL , κM , touching the conic
 171. section LGM , or the opposite hyperbolas L, M in the
 172. points L, M , meet one another in κ , and let the straight
 173. line κB meet the curve, or opposite curves, in the
 points G, B ; straight lines touching the curve or curves
 in G and B will either be parallel, or they will meet in
 the straight line LM .

For if the straight line κB bisect LM , by Cor. I.
 Prop. VI. Book III. κB is a diameter, and LM is a dou-
 ble ordinate to it; and, by Prop. II. Book III. tan-
 gents passing through G, B will be parallel to LM , and
 therefore parallel to one another. But let κB meet
 LM in H , and not bisect it. Let the tangents GN, BN
 be drawn, and meet one another in N , and if it be pos-
 sible let N not be in the straight line LM . Draw NL ,
 and let it meet κB in V . Let the tangents NG, NB
 meet the tangent κL in S and T . Then, by Prop. XV.

TK





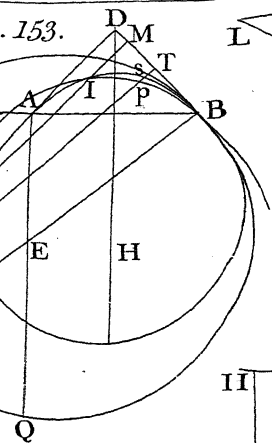


Fig. 159.

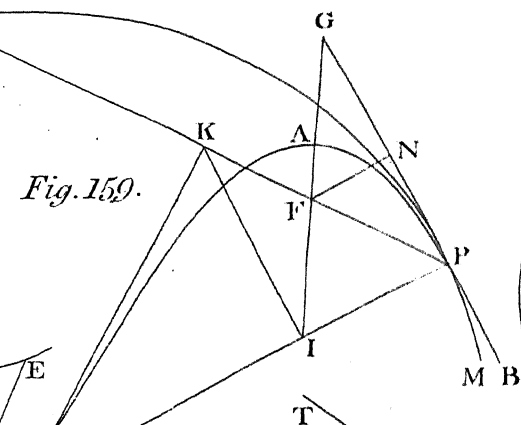


Fig. 155.

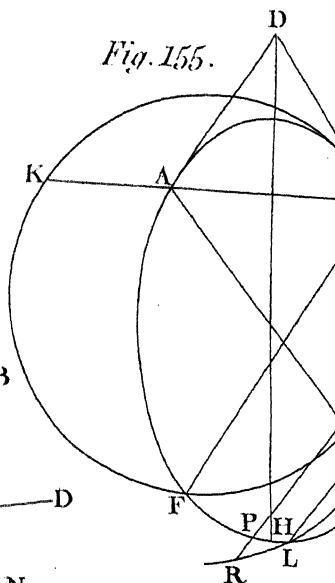


Fig. 156.

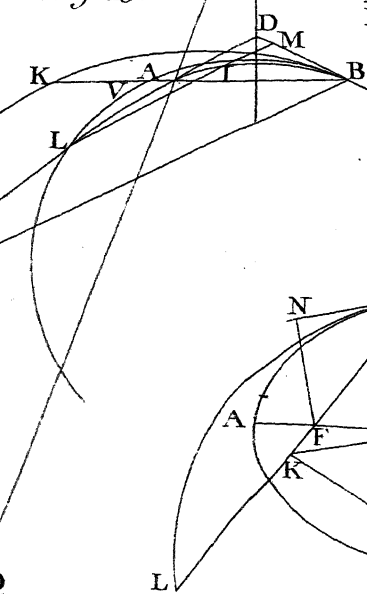


Fig. 158.

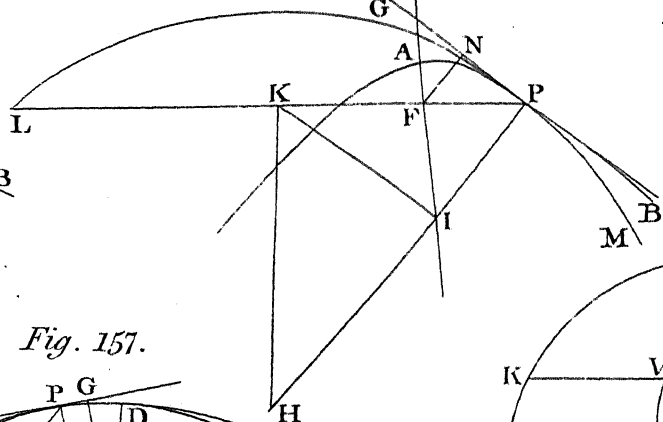


Fig. 157.

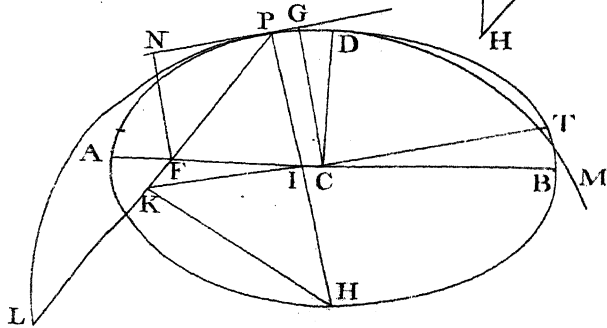
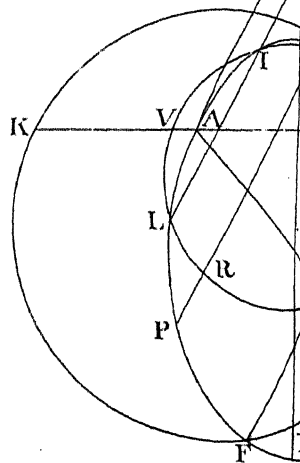
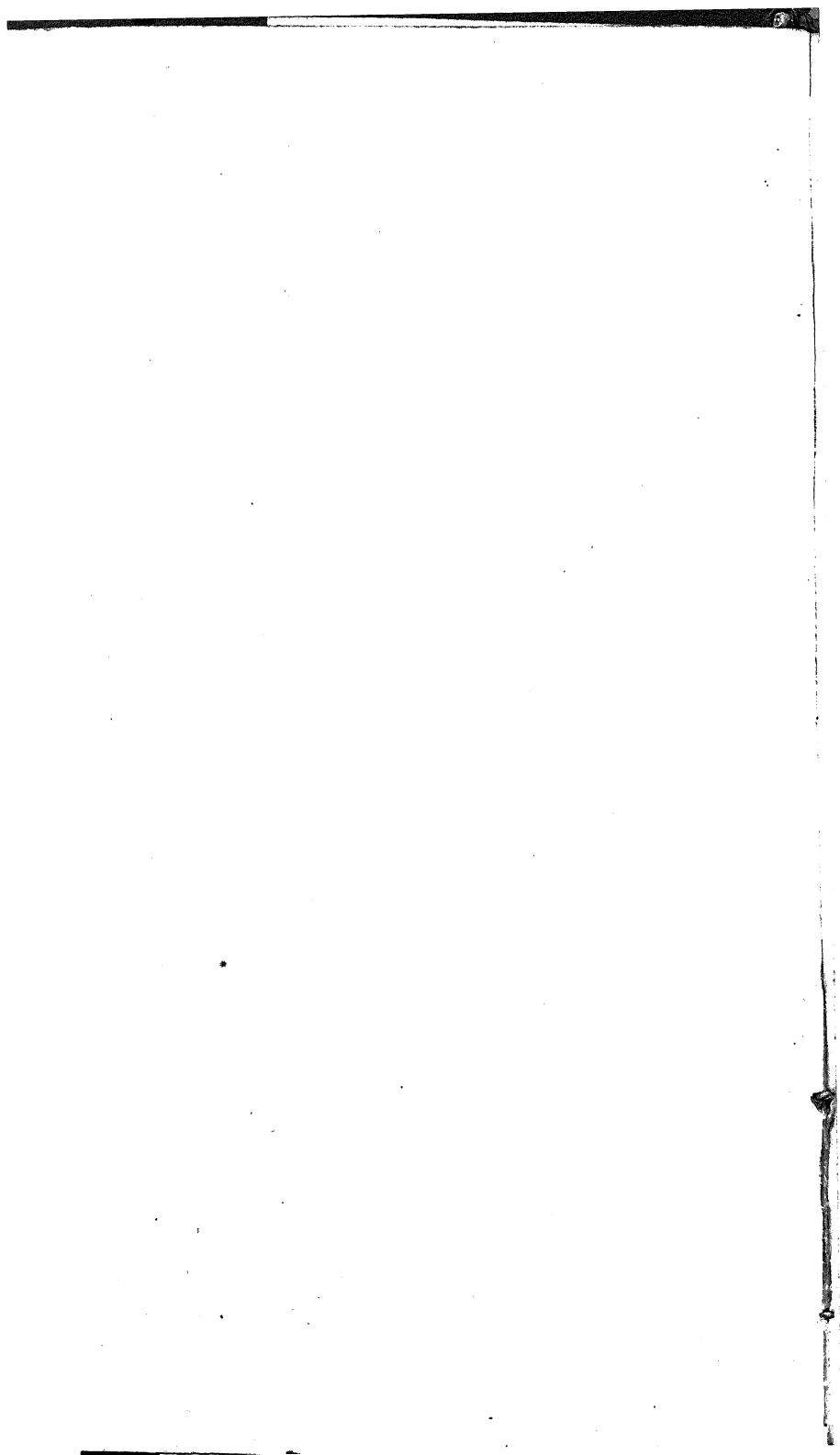


Fig.





$TK : KS :: TL : LS$; and therefore, on account of BOOK
IV.
 the harmonicals NK, NG, NV, NB , by the twelfth Lemma, $BK : KG :: BV : VG$. But, by Prop. XVI. $BK : KG :: BH : HG$; and therefore (11. v.) $BV : VG :: BH : HG$. Consequently (17. and 18. v.) $BG : VG :: BG : HG$, and (14. v.) VG is equal to HG ; which is absurd. The tangents GN, BN meet therefore in the straight line LM .

Cor. 1. If a straight line as LM , not passing through the center, cut a conic section, or opposite hyperbolas, in L, M , and from points N, D , &c. in LM , tangents NB, NG, DE, DF , &c. be drawn to the curve, or opposite curves, the straight lines BG, FE , &c. each joining the points of contact of two tangents drawn from the same point in LM , will meet one another in the point K , in which the tangents passing through L, M meet one another.

Cor. 2. If the straight lines BG, FE , not passing through the center, meet the curve, or opposite curves, in B, G and F, E , and one another in K , and if they be harmonically divided in B, H, G, K and F, Q, E, K ; then tangents passing through B, G , or through F, E , will meet one another in the straight line drawn through H, Q .

Cor. 3. If the tangents BN, GN meet in N , and the tangents FD, ED in D , and the straight lines BG, FE joining the points of contact meet one another in K , and the straight line ND in H and Q ; the straight lines KB, KF will be harmonically divided in K, G, H, B and in K, E, Q, F .

PROP. XVIII.

If the opposite sides of a trapezium inscribed in a conic section be not parallel, and neither pass through the center, the intersection of straight lines joining the opposite angular points, the intersection of two of the opposite sides, and

BOOK
IV.

and the intersections situated between these sides, of tangents passing through the points in which these sides meet the curve or curves, will be all four in the same straight line.

Fig. 164.

Let $G B F E$ be a trapezium inscribed in a conic section, or opposite hyperbolas, having no two of its opposite sides parallel, and no one of them passing through the center; the intersection I of the straight lines $D E, F G$ joining the opposite angular points, the intersection of the opposite sides $B F, G E$, and the intersections D, N , situated between $G V, B V$, of the tangents $E D, F D$, and $G N, B N$, are all four in the same straight line.

Let the sides $B G, F E$ meet one another in K ; and $N D$ being drawn, let it meet $K B$ in $M, K F$ in L , and $B F$ in V . Then, by Cor. 3. Prop. XVII. $K B$ is harmonically divided in K, G, M, B , and $K F$ in K, E, L, F . First, if the straight line $G E$ do not pass through V , draw $V K, V E$; and let $V E$ meet $K B$ in Q . Then, on account of the harmonicals $V K, V Q, V M, V B$, the straight line $K B$ will be harmonically divided in K, Q, M, B ; which is absurd, by Cor. 2. to the first Definition before Lemma XI. Consequently $B F, G E$ meet one another in the straight line $N D$. Secondly, if $N D$ do not pass through the point I , draw $I K, M I$, and let $M I$ meet $K F$ in P . Then, on account of the harmonicals $I K, G F, M P, B E$, the straight line $K F$ is harmonically divided, by Lemma XII. in K, E, P, F ; which, by Cor. 2. to the first Definition, before Lemma XI. is absurd. Consequently the straight lines $B E, G F$ meet one another in the straight line $N D$; and I, V, D, N the points of the intersections are in the same straight line $N D$.

SCHOLIUM.

In the following Problems, when the expression Art.
with

with a figure occurs, the article so numbered in the **BOOK** Scholium at the end of the third Book is referred to. **IV.**

The first six of the following solutions, although in several respects different, apply to the 59th, 60th, and 61st Problems in the *Arithmetica Universalis*, and also to the 22nd, 23rd, 24th, 25th, 26th, and 27th propositions in the first Book of the *Principia*.

PROP. XIX. PROB. I.

Given five points in the curve of a conic section, to describe the section.

Let E, A, B, C, D be five points given in the curve Fig. 163. of a conic section, to describe the section.

Draw AB, BD, DC, CA, BC; and through the point E draw EL parallel to AC, and ET parallel to AB. Let the straight line EL meet BC in L, and the straight line DC in H; and let ET meet BD in N. In ET, and on the same side of EH with N, take the segment ET so that HE may be to EN as LE to ET. Then ET being drawn it will touch the section, by Cor. 3. Prop. V. In the same way straight lines DF, AG may be drawn touching the section in D, A. Then if any two of the tangents, suppose DF, BT be parallel, the section will be an ellipse, according to Prop. VIII. Book I. and DB will be a diameter according to Cor. 1. Prop. II. Book II. In this case if a straight line be drawn from E parallel to DF, or BT, it will be an ordinate to BD; and the conjugate diameter to BD being found by art. 10. the section may be described.

But if no two of the tangents be parallel to one another, let BT meet DF in F, and AG in G. Draw FK bisecting BD in K, and GI bisecting AB in I. Then, by Cor. 1. Prop. VI. Book III. FK, GI will be dia-

BOOK diameters, and therefore if they are parallel, the section
 IV. will be a parabola: but if they are not parallel, they
 will meet in the center of the section.

If the section be a parabola let FK be bisected in P , and then it is evident, by Prop. V. Book III. and art. 9. that the section may be described. But if the section be not a parabola, let FK , GI meet in O , and O will be the center. Between FO , OK let a mean proportional OP be found, and, by Prop. VII. Book II. OP will be a femidiameter of the section, to which BD is a double ordinate. Consequently the diameter conjugate to OP being found by art 10. the section may be described.

PROP. XX.

Fig. 176. *If the straight lines AC , BD cutting a conic section in
 177. A , C and B , D , and meeting one another in G , meet in
 T and F a straight line TF which touches the section
 in E , the square of ET will be to the square of EF in
 the ratio compounded of the ratio of the rectangle under
 BG , GD to the rectangle under AG , GC and of the ra-
 tio of the rectangle under AT , TC , to the rectangle un-
 der BF , FD .*

∴ 176. First let the section be a parabola or hyperbola, and let the point T be on that side of the figure on which the section can be extended. Let the straight line IK be drawn parallel to BD , and let it meet the curve in I , K ; and let the square of the straight line vx be equal to the rectangle ITK . Then, by Prop. XIII. Book I. the square of ET is to the square of EF as the square of vx to the rectangle bfd . But the square of vx is to the rectangle bfd in the ratio compounded of the square of vx to the rectangle ATC and of the ratio of ATC to the rectangle bfd ; that

Fig. 160.

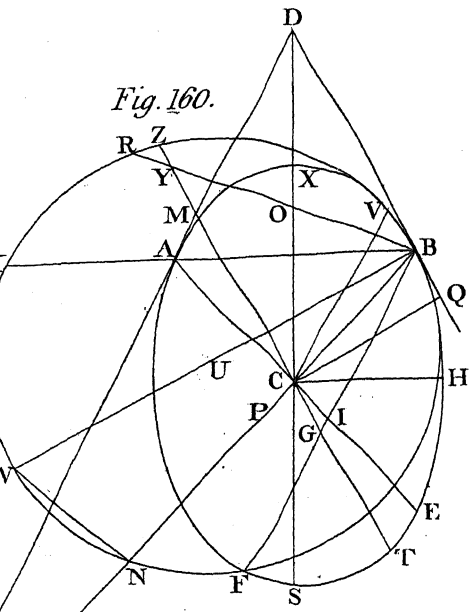


Fig. 161.

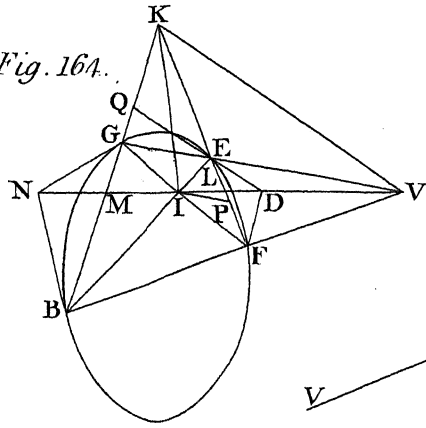


Fig. 162.

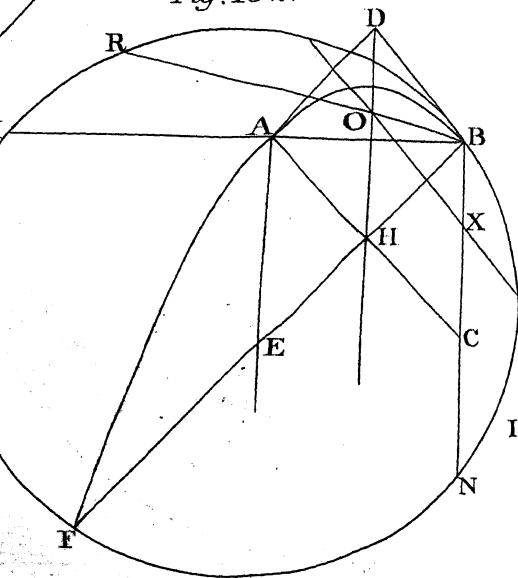


Fig. 163.

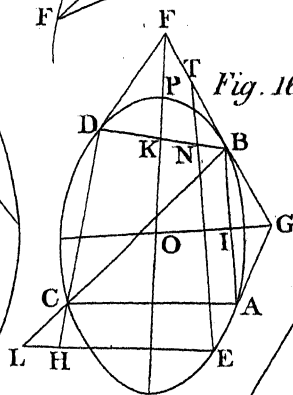
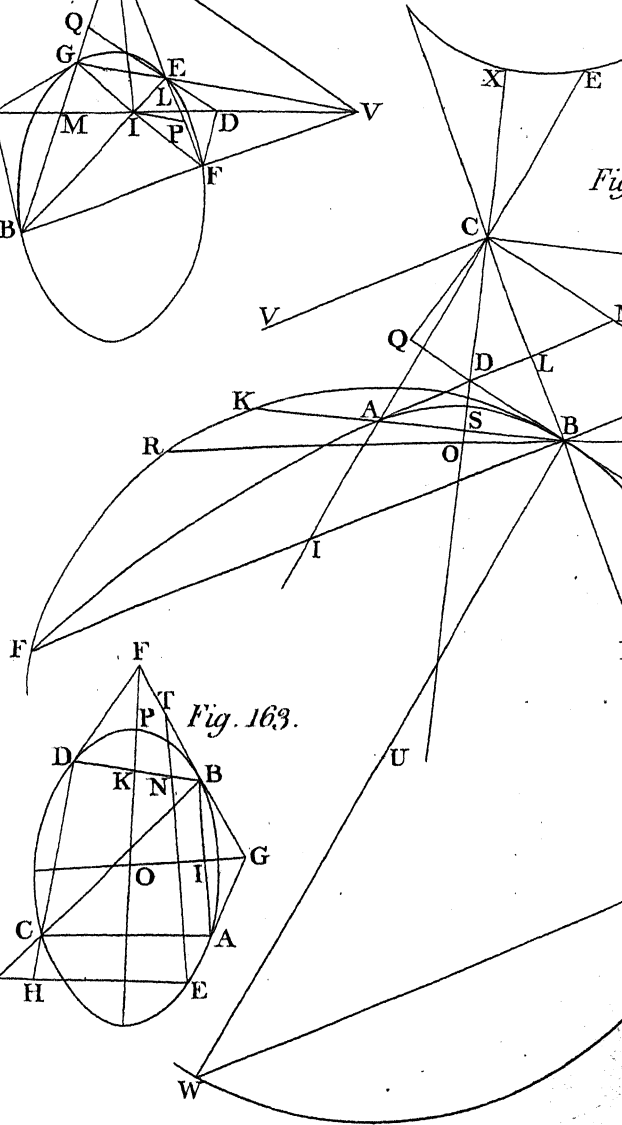
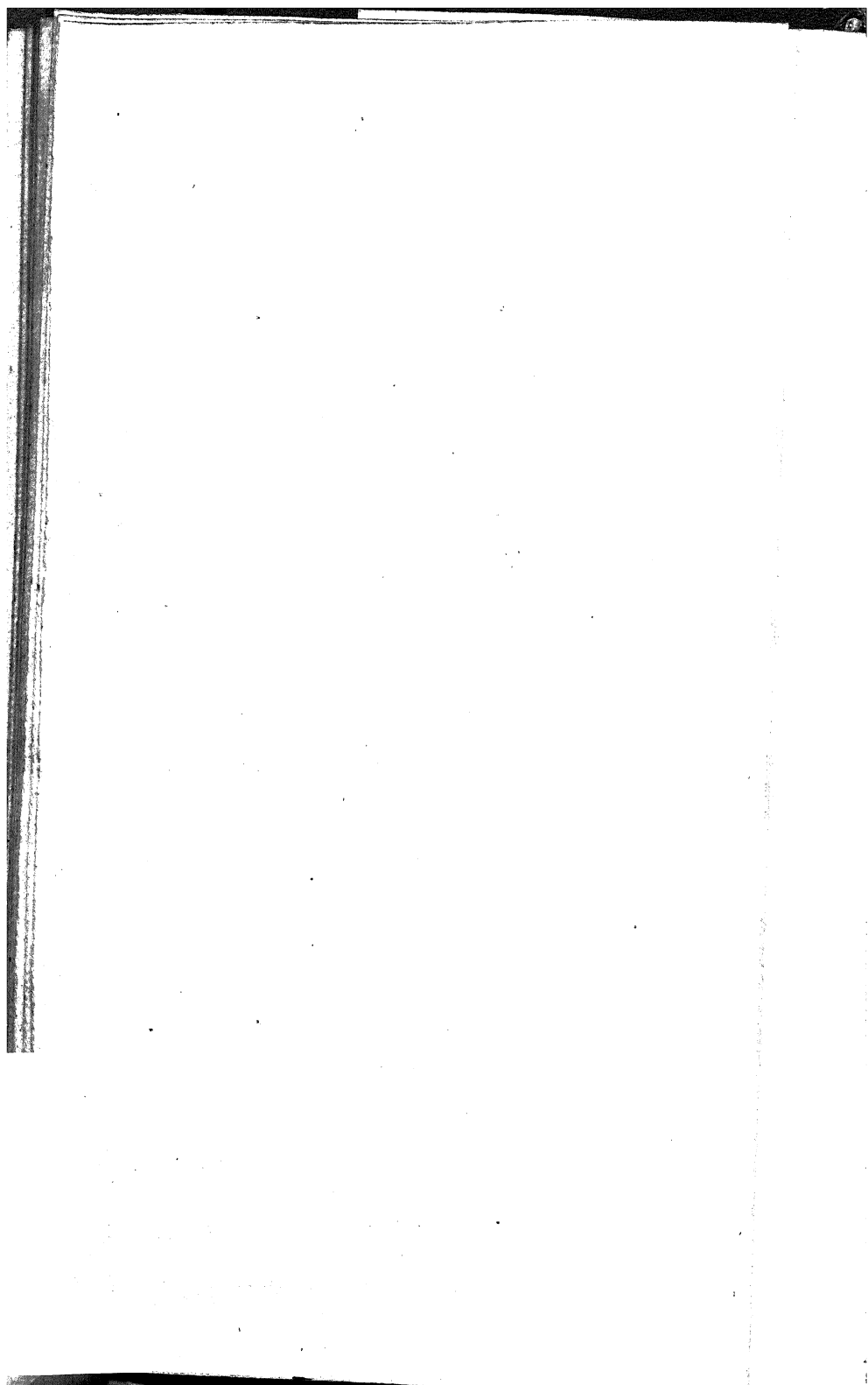


Fig.





that is, by Prop. XIII. Book I. in the ratio compounded of the ratio of the rectangle BGD to the rectangle AGC , and of the ratio of the rectangle ATC to the rectangle BFD . The square of ET therefore (II. v.) is to the square of EF in the ratio compounded of the ratio of the rectangle BGD to the rectangle AGC , and of the ratio of the rectangle ATC to the rectangle BFD .

Secondly, let the section be an ellipse, of which o is the center, and, the rest being as above, let oa be the semidiameter parallel to BD , od the semidiameter parallel to FT , and ob the semidiameter parallel to AC . Let it be $vx^2 : AT \times TC :: oa^2 : ob^2$; and then, as by Prop. V. Book II. $AT \times TC : ET^2 :: ob^2 : od^2$, we have

$$vx^2 : AT \times TC : ET^2 \\ oa^2 : ob^2 : od^2.$$

Consequently (22. v.) $vx^2 : ET^2 :: oa^2 : od^2$; and therefore by Prop. V. Book II. (and II. v.) $vx^2 : ET^2 :: BFXFD : EF^2$, and by alternation $ET^2 : EF^2 :: vx^2 : BFXFD$. Also, by the above, and Prop. V. Book II. (and II. v.) $vx^2 : AT \times TC :: BG \times GD : AG \times GC$. Again, the square of vx is to the rectangle under BF , FD in the ratio compounded of the ratio of the square of vx to the rectangle under AT , TC , that is, by the above, of the rectangle under BG , GD to the rectangle under AG , GC , and of the ratio of the rectangle under AT , TC to the rectangle under BF , FD . The square of ET therefore (II. v.) is to the square of EF in the ratio compounded of the ratio of the rectangle under BG , GD to the rectangle under AG , GC and of the ratio of the rectangle under AT , TC to the rectangle under BF , FD .

Cor. I. The next remaining as above, if the straight line TF meet in the point H the straight line HL , which

BOOK
IV.

Fig. 177.

Fig. 176
177

BOOK which touches the section in the point P and meets
 IV. TC in L , it may be demonstrated in the same manner that the square of ET is to the square of EH in the ratio compounded of the ratio of the square of PL to the rectangle under AL, LC , and of the ratio of the rectangle under AT, TC to the square of HR .

Cor. 2. If the squares of the straight lines MN, RS, TA, YZ be equal to the rectangles $BG \times GD, AG \times GC, AT \times TC, BF \times FD$, each to each; then ET will be to EF as the rectangle under MN, TA to the rectangle under YZ, RS . For, by the above, the rectangle under AG, GC is to the rectangle under BG, GD as the rectangle under AT, TC to the square of VX ; and therefore as the square of RS is to the square of MN , so is the square of AT to the square of VX . Consequently (22. vi.) as RS is to MN , so is TA to VX , and therefore (16. vi.) the rectangle under RS, VX is equal to the rectangle under MN, TA . Again, from the above, as the square of ET is to the square of EF , so is the square of VX to the rectangle under BF, FD , or the square of YZ ; and therefore, as ET to EF so is VX to YZ , that is (1. vi.) the rectangle under VX, RS to the rectangle under YZ, RS . Consequently, on account of the equal rectangles, ET is to EF as the rectangle under MN, TA to the rectangle under YZ, RS .

Cor. 3. Hence if any two straight lines AC, BD cut a conic section in A, C and B, D , and meet one another in G , and meet in T and F the straight line TF , which touches the section, the point of contact may be found. For let E be supposed to be the point of contact, and as above ET is to EF as the rectangle under a mean proportional between BG, GD and a mean proportional between AT, TC to a rectangle under a mean proportional between BF, FD and a
 mean

mean proportional between AG , GC . In the same way the point E may be found, if the tangent TF meet in H the straight line HL , which touches the section in P , and meets in L the straight line AC , which cuts the section in A , C , and meets TF in T .

BOOK
IV.

PROP. XXI. PROB. II.

Four points in the curve of a conic section, and a straight line touching the section, being given, let it be required to describe the section.

Let the four points A , B , D , C in the curve of a conic section, and the straight line TF touching the section, be given in position; it is required to describe the section.

Fig. 176.
177-
178.
179-

Case 1. Let the tangent TF pass through the point B , and draw AB , BD , DC , CA . First, let the opposite sides AC , BD of the trapezium be parallel, and let KR be drawn bisecting BD in K , and AC in R . Then KR is a diameter, by Cor. 2. Prop. II. Book III. and if it is parallel to the tangent FT , it will be the conjugate diameter to BD , by Cor. 2. Prop. IV. Book II. and the point K will be the center of the section. Consequently, if from the point A a straight line be drawn ordinately applied to the diameter BD , the magnitude of the diameter KR will be determined, by art. 10. and the section may then be described, by art. 8. But if the diameter KR be not parallel to the tangent FT , let them meet in F . Draw FD , and it will touch the section, by Cor. 2. Prop. VI. Book III. Through C draw the straight line CI parallel to the tangent FT , and let it meet the tangent FD in E , and BD in G . Let the segment EI be so taken in CG , that CE may be to EG as EG to EI ; and when the point G is without the section, let the point C be between E and I .

Fig. 178.

Then

BOOK Then the point I will be in the curve of the section,
IV. by Prop. XVII. Book I. and the five points A, B, D, C, I being in the curve, the section may be described as in Prop. XIX.

Secondly, let the opposite sides of the trapezium be
 Fig. 179. not parallel, and let AB, CD meet in L . Draw AD, CB , and let them meet in P . Through the points P, L draw the straight line PL , and let it meet the tangent TF in F , and draw FD . Then FD will touch the section, by Prop. XVIII. and, as above, a fifth point I may be found in the curve.

Fig. 176. Case 2. If the tangent TF do not pass through a
 177. given point, the point of contact may be found, by Cor. 3. Prop. XX. and the section may be described as in Prop. XIX.

PROP. XXII. PROB. III.

Three points being given in the curve of a conic section, and two straight lines touching it being given in position, let it be required to describe the section.

Case 1. If each of the two tangents pass through one of the given points, the points of contact will be given, and a straight line drawn through the third given point in the curve, and parallel to the straight line joining the points of contact, will meet the tangents. The segments of this straight line between the given point in the curve and the tangents will therefore be given; and if they are unequal, the line will be a secant, and the other point in which it cuts the curve may readily be obtained, according to Prop. IV. The solution may then be completed, by Prop. XXI. But if the segments are equal, the line will be a tangent, and in this case call it the third tangent, and draw a straight line from its point of contact to the point of
 con-

contact of the first tangent. Draw a straight line from the point in which the first and third tangents meet to the point of contact of the second tangent. Then, by Prop. XVI. this straight line will be harmonically divided in the point in which the first and third tangents meet, the point in which it meets the straight line joining their points of contact, the point of contact of the second tangent, and the other point in which it meets the curve. This other point may therefore be found by the second and third Corollaries in page 155, and the solution may be completed as above.

Case 2. Let the first tangent pass through one of the given points, but let the second tangent not pass through either of the other two; and, first, let the two tangents be parallel. Through the two other points draw a straight line, and if it be parallel to the tangents, a straight line drawn from the point of contact of the first tangent and bisecting this secant will meet the second tangent in its point of contact, as is evident from Prop. I. Book II. But if the straight line passing through the other two given points be not parallel to the tangents, it will meet them, and then, by Prop. XIII. Book I. the rectangle under its segments between the points and the first tangent will be to the square of the first tangent, as the rectangle under its segments between the given points and the second tangent to the square of the second tangent. The point of contact of the second tangent will therefore be obtained. Secondly, let DE be the first tangent passing through B one of the given points, and let A, C be the other two; and let DE the other tangent not be parallel to DE , but let them meet in D . Draw AC , and let it meet DE in K . Then if AC be parallel to one of the tangents, as in Fig. 180. let KE be a mean proportional between CK, KA , the point E being in DE ;

P

and

BOOK
IV.Fig. 180.
181.

BOOK IV. and a straight line drawn through B and E will meet DK in its point of contact, by Prop. XVII. Book I. But if the straight line AC be not parallel to either tangent, as in Fig. 181. let it meet DG in G , and DF in K ; and let GK be so divided, that the rectangle under AK, KC may be to the rectangle under CG, GA as the square of KE to the square of EG . Draw BE , and it will meet the tangent DF in the point of contact, by Cor. 2. Prop. XVII. Book I.

Fig. 182. Case 3. Let neither of the tangents KF or DG pass through a given point, and let A, B, C be the given points. Draw AC , and let it meet the tangent DG in D , and the tangent KF in K . Let KD be so divided in E , that the rectangle under AK, KC may be to the rectangle under CD, DA as the square of KE to the square of ED , and the straight line joining the points of contact will pass through E , by Cor. 2. Prop. XVII. Book I. Again, draw CB , and let it meet the tangent DG in L , and the tangent KF in I . Let LI be so divided in M , that the rectangle under CL, LB may be to the rectangle under BI, IC as the square of LM to the square of MI , and the straight line joining the points of contact will pass through M , by the same as above. Consequently the straight line ME will meet the tangents in the points of contact.

In every case, therefore, a section may be described, by Prop. XIX. or Prop. XXI.

Cor. If two tangents be given in position, and also two points in the curve of a conic section, but without the tangents, the point may be found in which the secant, passing through the given points, meets the straight line joining the points of contact, provided the secant be not parallel to the straight line joining the points of contact. For if the tangents and secant be parallel to one another, the point in which the secant is

is bisected will be the point required, as stated in the second case. In other cases the Cor. is evident from the above; for the tangents DG , DK being given in position, and the secant passing through the given points A , C , as in Fig. 180, 181, 182. the point E was ascertained.

BOOK
IV.

PROP. XXIII.

If three straight lines touch a conic section, a straight line drawn through the point in which the first and second tangents meet one another will be harmonically divided in this point of concurrence, in the point in which it meets the third tangent, and in the points in which it meets straight lines drawn from the first and second points of contact through the third point of contact.

Let the straight lines EF , EG , GH touch the conic section in the points F , C , H , the straight line EN , drawn through the point E , in which the first tangent EF and the second EG meet one another, is harmonically divided in E , in the point R in which it meets the third tangent GH , and in the points P , N in which it meets the straight lines FH , CH , drawn from the first and second points of contact F , C through the third point of contact H . Fig. 182.

For let EN meet the curve of the section in the points S , I ; and then, by Cor. 2. Prop. XVII. Book I.
 $SE \times EI : SR \times SI :: EP^2 : PR^2 :: EN^2 : NR^2$.
 Consequently (22. i.) $EP : PR :: EN : NR$.

PROP. XXIV. PROB. IV.

Two points being given in the curve of a conic section, and three straight lines being given in position and touching the curve, let it be required to describe the section.

BOOK IV. The two points A, B being given in the curve of a conic section, and the three straight lines CD, EF, GH being given in position and touching the section, let it be required to describe the section.

Fig. 183. Case 1. Let CD, GH two of the given tangents pass through the two given points A, B , and let them meet the other given tangent EF in D and G . If the tangent EF be parallel to AB , the straight line joining the points of contact, and GD be bisected in E , the point E will be that in which GD touches the section, by Prop. IV. But if GD, AB be not parallel, let them meet in F , and let GD be so divided that DF may be to FG as DE to EG , and E will be the point in which GD touches the section, by Prop. XV. Three points will therefore be obtained in the curve, and consequently the solution may be completed, by Prop. XXII.

Fig. 184. Case 2. Let GH one of the given tangents pass through
185. B one of the given points, and meet the given tangent EF in G . Let the straight line AB be drawn, and let it meet the given tangent CD in D ; and if AB be pa-

Fig. 184. rallel to EF , let the segment DL be taken in BD a mean proportional between BD, DA , and the straight line joining the points in which EF, CD touch the section, will pass through the point L , by Prop. XVII. Book I. Draw GL , and let it meet the tangent CD in K ; and if the tangents GH, CD be parallel, let LG be to LK as BG to KC , and C will be the point in which CD touches the section. For let GK meet the curve of the section in P and N , and the rectangle under PG, GN is to the rectangle under NK, KP as the square of GL to the square of LK , by Prop. XVII. Book I. and the square of BG is to the square of CD , the segment between the point K and the point of contact, in the same ratio, by Prop. XIII. Book I. If the straight line AB be not parallel to the tangent EF , the point

point L may be found by Cor. 2. Prop. XVII. Book I. BOOK
IV.
and the tangents GH , CD being parallel, the point of contact c may be found as above.

But if the tangent GH meet the tangent EF in G , Fig. 185.
and the tangent CD in H , let GH be so divided in N ,
that HB may be to BG as HN to GN , and the straight
line joining the points of contact of the tangents EF ,
 CD will pass through N , by Prop. XV. Again, let the
straight line AB meet the tangent EF in P , and the
tangent CD in D , and as the rectangle under AP , PB
is to the rectangle under BD , DA , so let the square of
 PL be to the square of DL , and the straight line join-
ing the points of contact of the tangents EF , CD will
pass through L , by Cor. 2. Prop. XVII. Book I. Let
the straight line LN therefore be drawn, and it will
meet the tangents EF , CD in the points of contact.
If the straight line AB be parallel to the tangent FG ,
the point L may be found, by Prop. XVII. Book I. as
above.

Case 3. Let the points A , B be without any one of Fig. 186.
the tangents, and let the straight lines AB , EF , GH be
parallel. Let the tangent CD meet the tangent GH
in G , and the tangent EF in E ; and let the straight
line AB meet the tangent CD in D . In AD let the seg-
ment DL be taken a mean proportional between AD ,
 DB , and the straight line joining the points of contact
of the tangents CD , EF will pass through the point L ,
by Prop. XVII. Book I. Let the straight line GLN
be drawn; let it meet the tangent EF in K , and let GL
be to LK as GN to NK , and the straight line joining
the points of contact of the tangents GH , EF will pass
through N , by Prop. XXIII. Again, let AP be bi-
sected in P , and the straight line joining the points of
contact of the tangents GH , EF will pass through P ,
by Prop. I. Book II. Consequently if the straight line

BOOK NP be drawn, it will meet the tangents in the points of
IV. contact.

Fig. 187.

Case 4. Let the tangent CD meet the tangent GH in G and EF in E , and let the straight line AB be parallel to the tangent CD , and meet the tangent GH in K , and the tangent EF in M . In the straight line KM let the segment MP be taken a mean proportional between AM , MB ; and let the segment KL be a mean proportional between BK , KA ; and the straight line joining the points of contact of the tangents CD , EF will pass through the point P , but the straight line joining the points of contact of the tangents CD , GH will pass through the point L , by Prop. XVII. Book I. Let the straight line GP be drawn, and meet the tangent EF in I , and as GP to PI so let GN be to NI . Again, let LE be drawn, and let it meet the tangent GH in Q , and let QL be to LE as QR to RE . Then, by Prop. XXIII. the straight line NR passes through the points of contact of the tangents EF , GH .

Fig. 188.

Case 5. Let the straight line AB be not parallel to any one of the tangents GH , DC , EF , and let it meet the tangents DC , EF in E , the point in which they meet one another. Let the straight line AB be divided in the point L , so that BE may be to EA as BL to LA , and the straight line joining the points of contact of the tangents DC , EF will pass through the point L , by Prop. XVI. Let the straight line GL be drawn, and meet the tangent EF in K , and in the straight line GL let the segment GN be so taken, that GL may be to LK as GN to NK ; and, by Prop. XXIII. the straight line joining the points of contact of the tangents EF , GH passes through the point N . Again, let the straight line AB meet the tangent GH in M , and in AB let the segment AR be so taken that the rectangle AEB may be to the rectangle AMB as the square of ER to the square

square of RM , and the straight line joining the points of contact of the tangents EF , GH will pass through the point R , by Prop. XVII. Book I. Consequently the straight line NR will meet the tangents EF , GH in the points of contact.

Case 6. Let the tangent CD meet the tangent GH in G , and the tangent EF in E . Let the straight line AB be drawn, and let it meet the tangent GH in M , the tangent EF in K , and the tangent CD in D . Let the straight line AB be so divided in L and N , that the rectangle under AK , KB may be to the rectangle under BM , MA as the square of KL to the square of ML ; and as the rectangle under AD , DB to the rectangle under BM , MA , so let the square of DN be to the square of NM . Then the straight line joining the points of contact of the tangents EF , GH will pass through the point L , but the straight line joining the points of contact of the tangents CD , GH will pass through N , by Prop. XVII. Book I. Let the straight line NE be drawn, and meet the tangent GH in R ; and let NE be so divided in P , that EN may be to NR as EP to PR ; and, by Prop. XXIII. the straight line joining the points of contact of the tangents EF , GH passes through P . Consequently if the straight line LP be drawn, it will meet the tangents EF , GH in the points of contact.

In every case therefore the section may be described by Prop. XIX. as five points may be easily found.

PROP. XXV.

If the four straight lines AE , EG , GH , HD touch a conic section in A , B , C , D , and meet one another in E , G , F , H ; and if the straight lines AC , BD be drawn joining the opposite points A , C and B , D and meeting one another; the straight line FG drawn through the oppo-

Fig. 174.

BOOK
IV.

Site points in which the tangents meet one another will pass through the point in which the straight lines AC , BD meet one another.

For let the straight line FG meet the section or opposite sections in N , M and the straight line AC in I , and, by Cor. 2. Prop. XVII. Book I. the rectangle under NG , GM is to the rectangle under NF , FM as the square of GI to the square of FI . But, if it be possible, let the straight line FG meet the straight line BD in P , and, as above, the rectangle under NG , GM is to the rectangle under NF , FM as the square of PG to the square of FP . Consequently (II. v.) as the square of GI to the square of FI , so is the square of PG to the square of FP ; and therefore (22. vi.) as GI to FI so is PG to FP . Consequently (18. v.) as GI to FG so is GP to FG , and therefore (14. v.) the straight lines GI , GP are equal; which is absurd.

The rest remaining, if the straight line FG be a diameter of a parabola, or parallel to an asymptote of an hyperbola, the square of NI , and also the square of NP , will be equal to the rectangle under GN , NF , by Prop. XXIII. Book III. and therefore GI , GP are equal; which is absurd.

In any case therefore the straight lines AC , BD , FG meet one another in I , and if the straight line EH be drawn joining the remaining opposite points, in which the tangents meet one another, it will pass through the point I , for the same reasons as above.

Cor. Hence, if four straight lines AE , EG , GH , HD touch a conic section and meet one other in E , G , F , H , and if the straight lines FG , EH joining the opposite points meet one another in I ; the straight lines AC , BD joining the opposite points of contact will pass through the point I .

PROP.

PROP. XXVI. PROB. V.

BOOK
IV.

The point A being given in the curve of a conic section, and the four straight lines EF, EG, GH, HD touching the section being given in position, let it be required to describe the section.

Fig. 174.
175.

Let the tangents meet one another in E, F, G, H, and let the straight lines FG, EH be drawn, joining the opposite points F, G and E, H, and meeting one another in I. Let the straight line AI be drawn, and let it meet the opposite tangent GH in C, and if the point A be in one of the tangents, the straight line GH will touch the section in C, as is evident from Prop. XXV. But if the point A be not in one of the tangents, let AC meet EF in K, and let the straight line KC be so divided in R that the square of KI may be to the square of IC as the rectangle AKR to the rectangle RCA; and as by the last Prop. the straight line joining the points of contact of EF, GH passes through I, the point R will be in the curve of the section, by Cor. 2. Prop. XVII. Book I. Consequently in any case the section may be described, by Prop. XXIV.

PROP. XXVII. PROB. VI.

Five straight lines being given in position and touching a conic section, let it be required to find the points in which they touch the section.

Let the straight lines AB, BC, CD, DE, EA touch a conic section, and let it be required to find the points of contact in them.

Fig. 190.

Let ABCDEA be the quinquelateral figure contained by the tangents, and let AB be called the first side, BC the second, &c. and let FBCD be the quadrilateral figure contained under the four first sides, and draw the diagonals BD, FC meeting one another in M.

The

BOOK IV. The first side AB of the quinquelateral figure being now omitted, let $ICDE$ be a quadrilateral contained by the others, and let ID , CE the diagonals be drawn meeting one another in N . Then MN being drawn, it will pass through the points in which the second side BC and the fourth DE touch the section, by Cor. Prop. XXV. In the same manner the points may be found in which AB , CD , AE touch the section, and therefore the section may be described, by Prop. XIX.

PROP. XXVIII.

Fig. 191. Let ED be an equilateral hyperbola, of which AF , AC are the asymptotes, and let it cut in the point D the curve of the parabola AD , of which AF is the axis, and the segment AF equal to the parameter of the axis; let there be drawn to the curve of the hyperbola the straight line FE parallel to the asymptote AC , and from the point D , in which the curves of the hyperbola and parabola cut one another, let there be drawn to the asymptote AF the straight line DB parallel to the asymptote AC ; then will the straight lines BD , AB be two mean proportionals between AF , FE .

For as ED is an equilateral hyperbola, the angle AFE is a right one, by Prop. XVI. Book III. (and 29. i.) The straight line DB is therefore an ordinate to the axis of the parabola, and, by Prop. III. Book III. (and 17. vi.) $AF : BD :: BD : AB$. Again, by Cor. 2. Prop. XVII. Book III. $AF : AB :: BD : FE$, and therefore by alternation $AF : BD :: AB : FE$. Consequently (II. v.) $AF : BD :: BD : AB$, and $BD : AB :: AB : FE$.

Cor. Hence if two straight lines as AF , FE be given, two mean proportionals may be found between them. For let the two straight lines AF , FE be at right angles

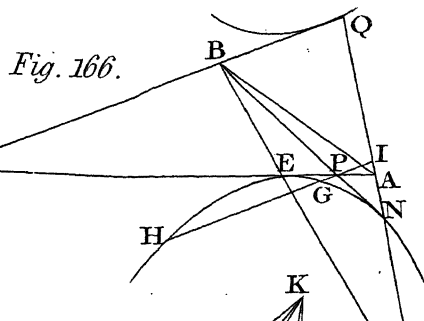
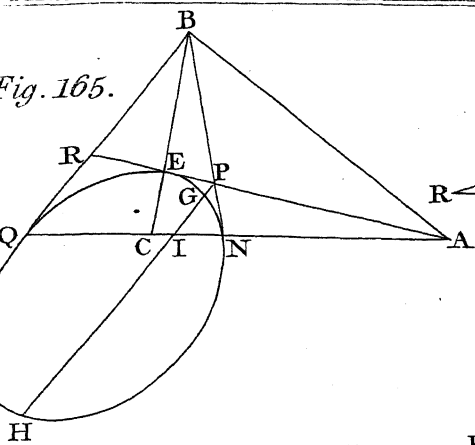


Fig. 171.

Fig. 170.

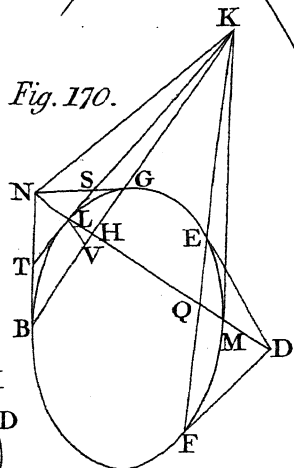


Fig. 175.

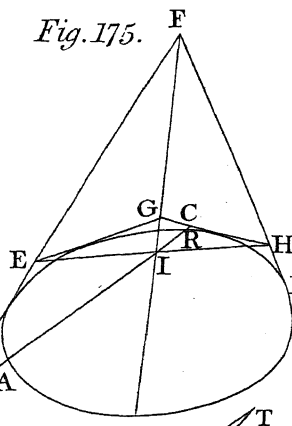


Fig. 169.

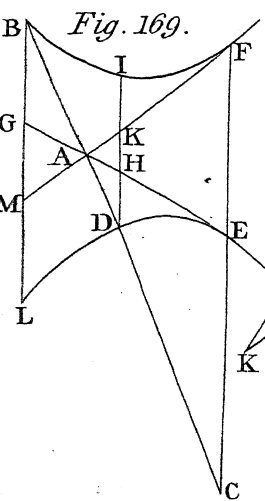


Fig. 167.

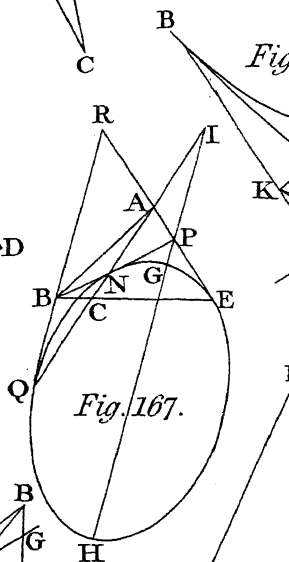


Fig. 168.

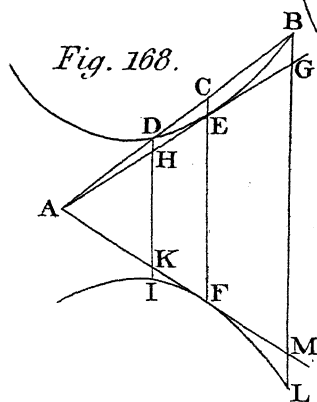
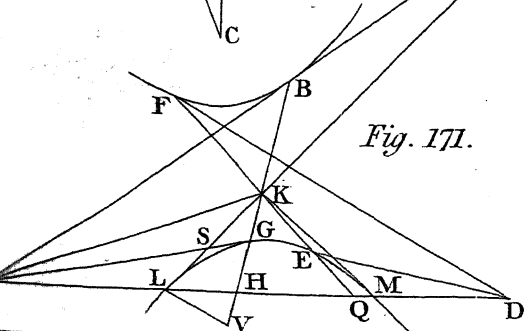
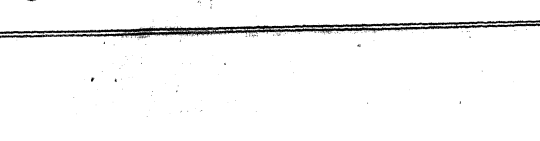
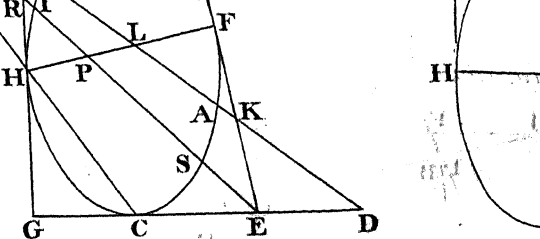
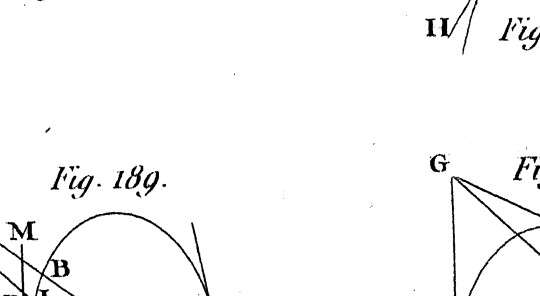
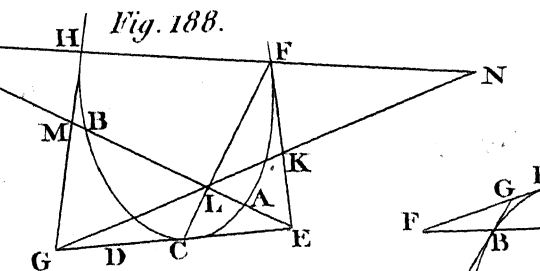
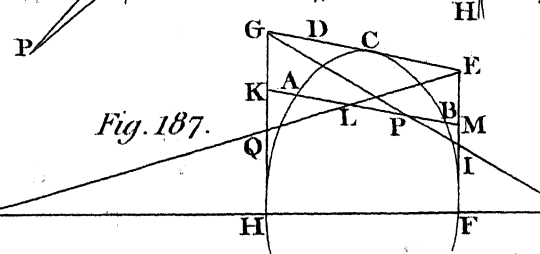
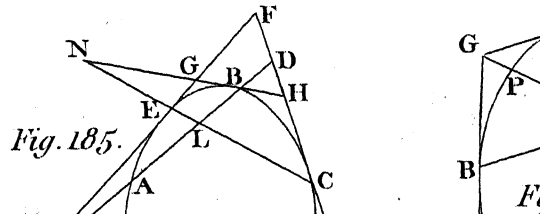
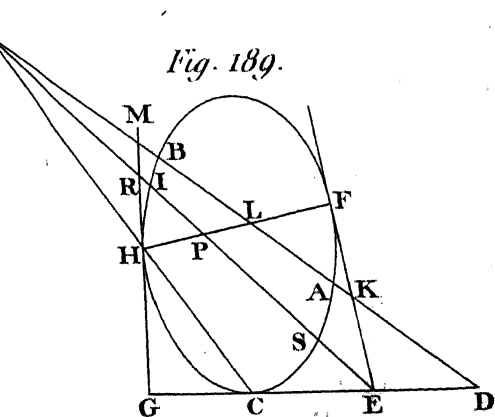
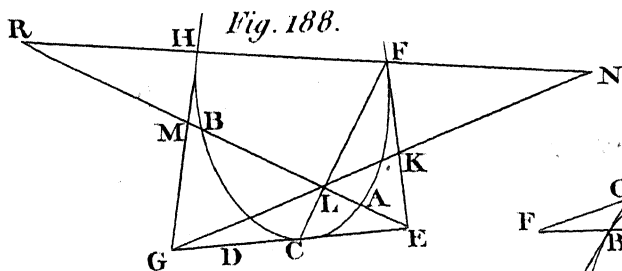
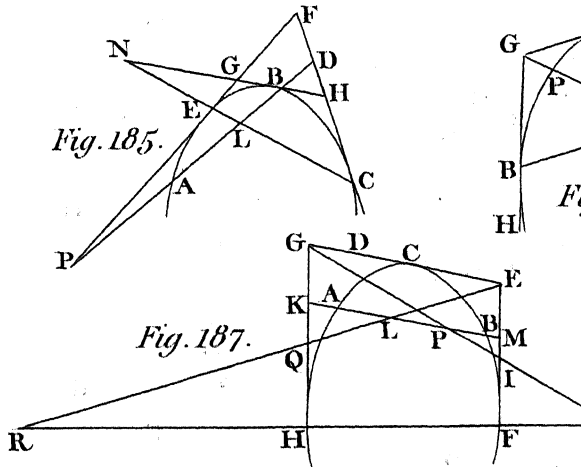
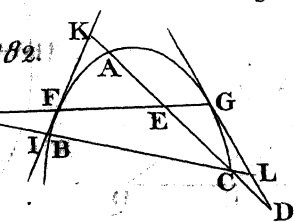
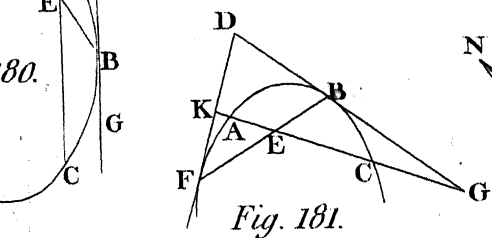
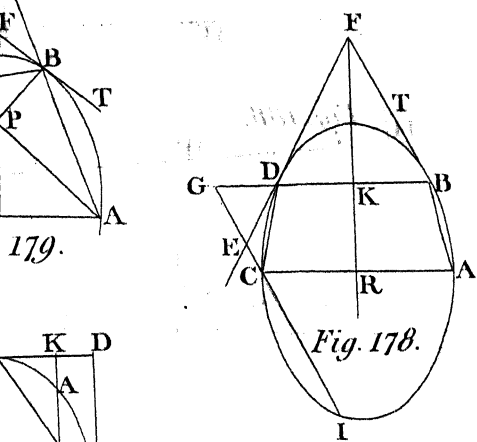
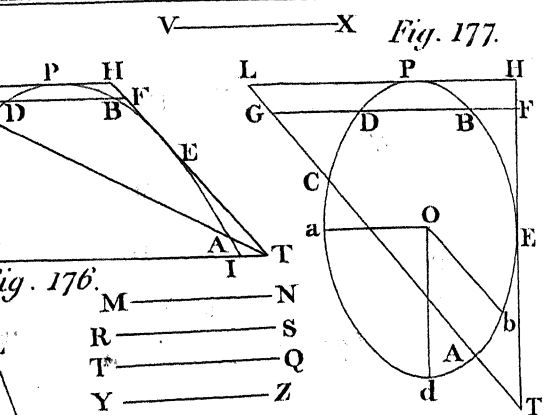
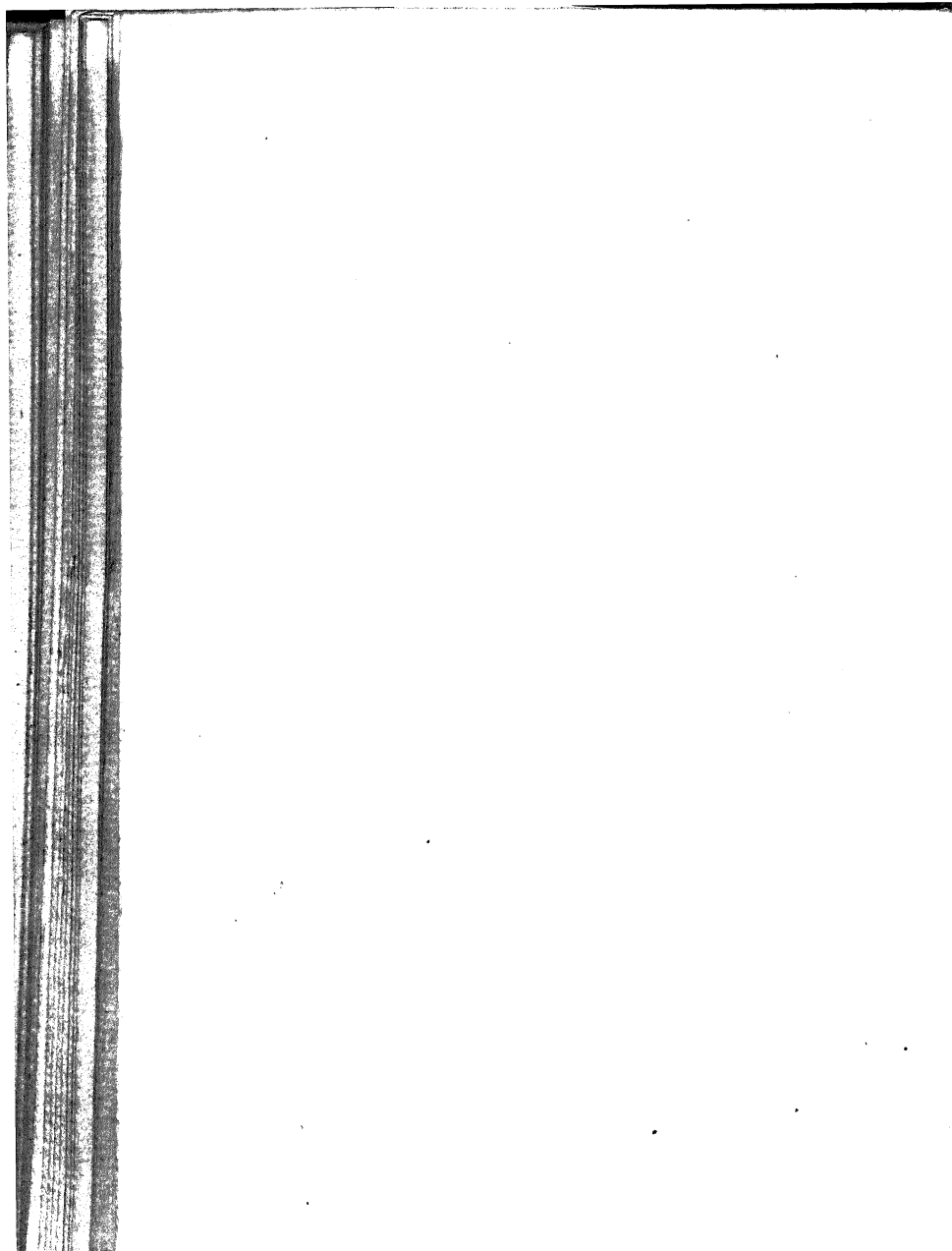


Fig. 171.







gles to one another, and let the parallelogram $A F E G$ be completed. Let the parabola $A D$ be described, of which $A F$ is the axis, and the segment $A F$ equal to its parameter. Again, let an equilateral hyperbola be described through the point E , of which $A F$, $A G$ are the asymptotes, and let its curve cut the curve of the parabola in D . Let $D B$ be drawn to $A F$ and parallel to $A G$, and let $D C$ be drawn to $A G$ parallel to $A F$. Then the straight lines $B D$, $A B$ will be two mean proportionals between $A F$, $F E$.

PROP. XXIX.

Let $A E$ be a parabola, of which $A D$ is the axis, and $A B$ a segment in it equal to half its parameter; let the straight line $B G$ be perpendicular to the axis, and draw $A G$; with the center G and distance $G A$ describe the circle $A C E$ cutting the axis in the point C and the curve of the parabola in E , and let $E D$ be drawn an ordinate to the axis; the straight lines $E D$, $A D$ will be two mean proportionals between $A C$ and a straight line equal to the double of $G B$. Fig. 192.

For, by the construction, (and 3. iii.) the straight line $A C$ is equal to the parameter of the axis, and therefore, by Prop. III. Book III. the square of $D E$ is equal to the rectangle under $A C$, $A D$. Let $D E$ meet the circumference of the circle again in F , and let the segment $F H$ be equal to the segment $D E$. Then the rectangle $E D F$ will be equal to the rectangle $A D C$, (35. iii.) and therefore the square of $D E$ together with the rectangle $E D F$ are equal to the rectangles $D A C$, $A D C$ together, that is, (2. ii.) to the square of $A D$. But the square of $D E$ together with the rectangle $E D F$ is equal to the rectangle under $D E$, and a straight line equal to the sum of $E D$, $D F$ (1. ii.); and therefore

BOOK fore the square of DE together with the rectangle EDF
IV. is equal to the rectangle under DE , HD . Consequently (17. vi.) $DE : AD :: AD : HD$; and by the above $AC : DE :: DE : AD$. But DH is double of GB ; for let GI be drawn parallel to AD , and let it meet DH in I . Then GB , ID (34. i.) are equal to one another, as are also (3. iii.) EI , IF to one another, and therefore HI is equal to ID . The Proposition is therefore evident.

Cor. Hence, by means of a parabola and a circle, a method is evident of finding two mean proportionals between two given straight lines.

PROP. XXX.

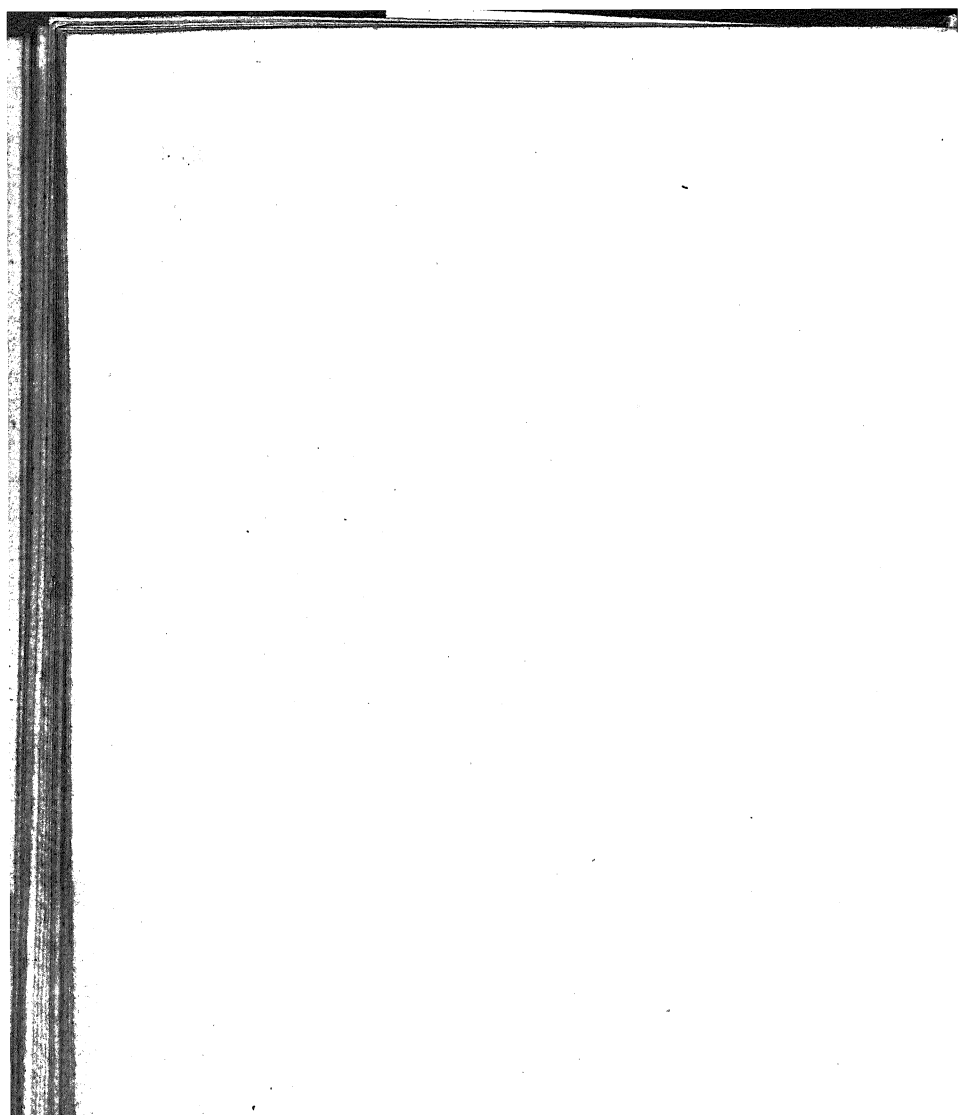
Fig. 193. *From any point B in the curve of the equilateral hyperbola BE let the straight lines BA, BD be drawn to the asymptotes CA, CD, and let BA be parallel to CD and BD parallel to CA, and let AD the diameter of the parallelogram be drawn; with the center B and a distance equal to the double of AD let a circle be described, and let it meet the curve of the hyperbola in E; from E draw EF to the asymptote CD and parallel to CA; then, AF being drawn, the angle BAF will be a third part of the angle BAD.*

For let AF meet BD in G . Bisect DF in K , and draw KI parallel to BD , and let it meet AF in the point I . Draw PI . Then as the hyperbola is equilateral, the angle ACD is a right one, and therefore (29. i.) each of the angles FKI , DKI is a right one, and (4. i.) FI , ID are equal. But, on account of the equals FK , KD and the parallels KI , DG , FI is equal to IG . Again, (15. and 29. i.) the triangles ABG , FCA are equiangular, and therefore (4. vi.) $AB : BG :: CF : CA$, and (16. vi.) the rectangle under BG , CF is equal

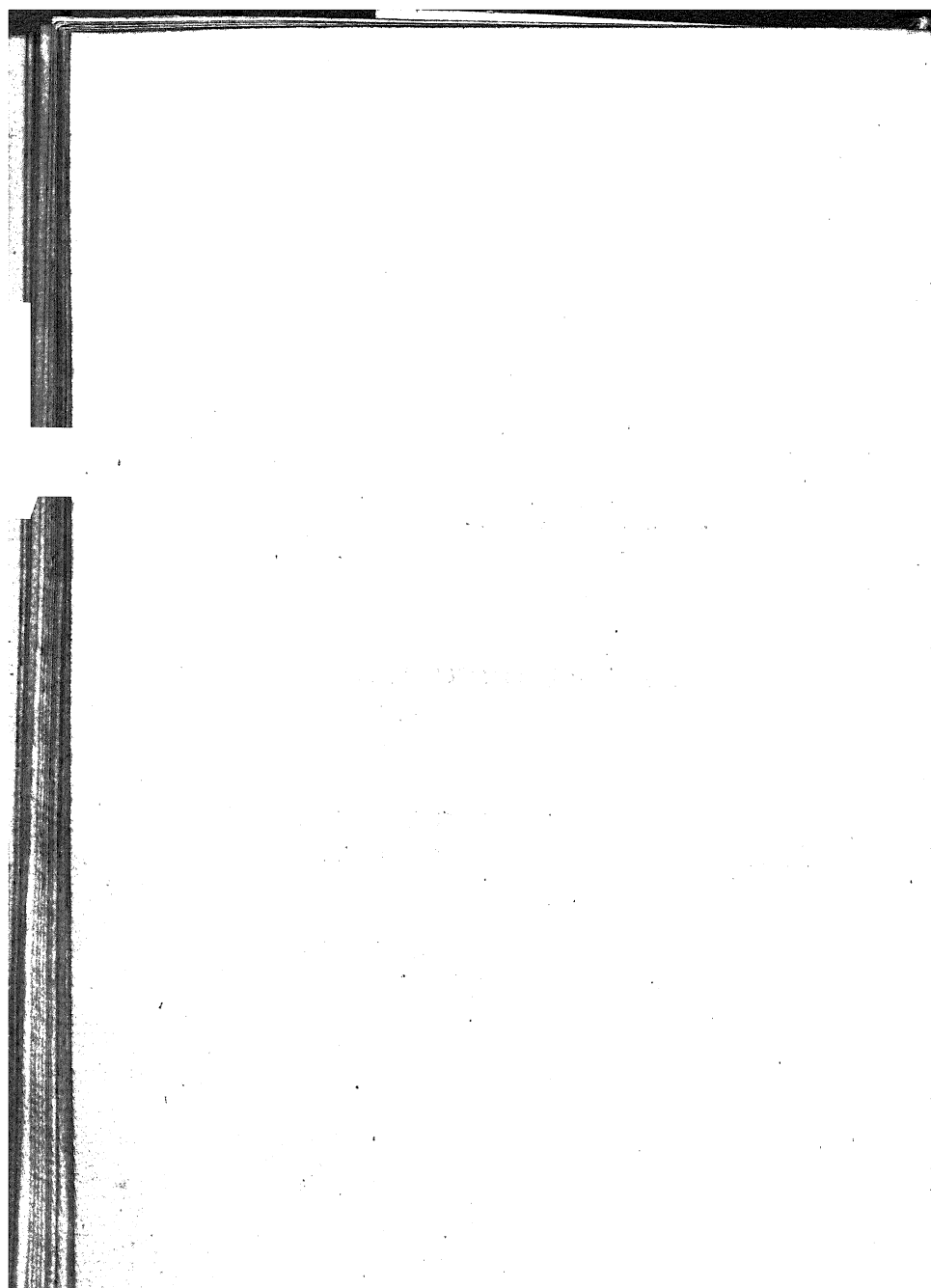
to

to the rectangle under BA, AC . But, by Prop. XVII. BOOK
IV.
 Book III. the rectangle under BA, AC is equal to the
 rectangle under EF, CF , and therefore the rectangle
 under BG, CF is equal to the rectangle under EF, CF .
 Consequently BG is equal to EF , and therefore (33. i.)
 BE is equal to GF ; and therefore, by the construction,
 FI, ID, DA are equal. The angles DFI, FDI are
 therefore equal to one another, as are also the angles
 DIA, DAI to one another. The angle DAI is there-
 fore equal to the double (32. i.) of DFI , or of its
 equal the angle BAG . Consequently the angle BAG
 is equal to a third part of the angle BAD .

Cor. Hence, by means of an equilateral hyperbola
 and its asymptotes, an angle may be divided into three
 equal parts.



A
TREATISE
ON THE
PRIMARY PROPERTIES
OF
CONCHOIDS, THE CISSOID, THE QUADRATRIX,
CYCLOIDS, THE LOGARITHMIC CURVE,
AND
THE LOGARITHMIC, ARCHIMEDEAN, AND
HYPERBOLIC SPIRALS.



A
TREATISE
ON THE
PRIMARY PROPERTIES
OF
CONCHOIDS, THE CISSOID, THE QUADRATRIX,
CYCLOIDS, THE LOGARITHMIC CURVE,
AND THE
LOGARITHMIC, ARCHIMEDEAN, AND HYPERBOLIC SPIRALS.

SECTION I.
Of Conchoids.

DEFINITIONS.

I.

[F the fixed point P be without the straight line TRX , and if the straight line DL of indefinite length pass through P , and D, A be two fixed points in DL ; then if the straight line DL , always passing through P , be moved in such a manner that the point A is always in TRX , and the point D describe the curve DVG , the curve DVG is called a *Conchoid*. Plate XXVI.
Fig. 1.
2.
3.

II.

The fixed point P is called the *Pole* or *Center* of the conchoid, and the straight line TRX is called the *Directrix* of the conchoid.

a

III.

SECT.
I.

III.

The straight line PRV perpendicular to the directrix, meeting it in R and the curve in V , is called the *Axis* of the conchoid; and the point V is called the *Vertex* of the axis.

IV.

If the directrix be between the curve and the pole, as in Fig. 1. the curve is called the *Superior Conchoid*, or the *Conchoid of Nicomedes*.

V.

If the curve be between the directrix and the pole, and the segment RV of the axis be less than the segment PR , as in Fig. 2. the curve is called the *Inferior Conchoid*.

VI.

If the curve and pole be on the same side of the directrix, and the segment VR of the axis be greater than the segment PR , as in Fig. 3. the curve $DPVG$ is called the *Nodated Conchoid*.

Corollary to the preceding Definitions. In any one of the three conchoids if a straight line pass through the pole, and cut the directrix and curve, its segment intercepted between the directrix and the curve will be equal to the segment RV of the axis, between the directrix and curve. For the points D, A being fixed in the straight line DL , the magnitude of the segment DA is the same in every position of the moving line DL ; and when DL falls upon the axis the describing point D coincides with V , and the point A coincides with R .

VII.

A straight line drawn from any point in the conchoid perpendicular to the axis is called an *Ordinate* to the axis.

PROP. I.

SECT.
I.

The conchoid and its directrix being produced, on either side of the axis, continually approach nearer and nearer to one another, but never meet.

For, the rest remaining as in the Definitions of each of the conchoids, let the straight line DB be perpendicular to the directrix TX . Then PR , DB being perpendicular to the directrix TX , the triangles (15. and 29. i.) APR , ADB are equiangular; and therefore (4. vi.) $AP : PR :: AD : DB$. Consequently, as by the Cor. to the Definitions AD is equal to RV , the rectangle (16. vi.) under AP , DB is equal to the rectangle under PR , RV . But DB is the distance of the curve at the point D from the directrix; and it is evident that AP increases as the distance of D from the axis increases. The distance of the describing point D , therefore, from the directrix must decrease as D recedes from the vertex, as PR , RV are constant; and as the rectangle under AP , DB is a constant magnitude, the points D and B cannot coincide. Hence the Proposition is evident.

Fig. 1.
2.
3.

Cor. The directrix is also an asymptote to the conchoid.

PROP. II.

If an ordinate be drawn from any point in either of the conchoids to the axis, a straight line drawn from the pole to the same point in the curve will be a fourth proportional to the distance of the ordinate from the directrix, the distance of the vertex from the directrix, and the distance of the ordinate from the pole.

The rest remaining as in the preceding Proposition,

Fig. 1.
2.
3.

SECT. I. and the six first Definitions, let DE be an ordinate to the axis according to the seventh Definition, and let it meet the axis in E ; and then RE is to RV as PE to PD .

For BE is a parallelogram, and therefore (34. i.) RE is equal to BD , and, by the Cor. to the Definitions AD is equal to RV . Consequently $RE : RV :: BD : AD$. But the triangles (29. i.) $BD A$, $EP D$ are equiangular, and therefore (4. vi.) $BD : AD :: PE : PD$. Consequently (11. v.) $RE : RV :: PE : PD$.

Cor. 1. The rect remaining as above, with R as a center and RV as a distance describe $VI C$ a quadrant of a circle, and let it cut DE , or DE produced, in I , and draw IR . Then IR is equal to RV , and therefore by the above $RE : IR :: PE : PD$. In the superior conchoid the angle at E is common to the two triangles REI , $PE D$, and in the other two conchoids, the angle at E in the triangle REI is equal to the angle at E in the triangle $PE D$, each of them being a right angle. Consequently (7. vi.) the triangles REI , $PE D$ are equiangular, and therefore (4. vi.) $PE : ED :: RE : IE$.

Cor. 2. From the preceding Cor. the equation of each of the conchoids may be easily deduced. For in each of them put $RV = a$, $PR = b$, $RE = x$, and $DE = y$.

Fig. 1. 1. In the superior conchoid (47. i.) $IE = \sqrt{a^2 - x^2}$, as IR , RV are equal; and $PE = b + x$. Consequently $b + x : y :: x : \frac{xy}{b + x} = IE = \sqrt{a^2 - x^2}$; and $y = \frac{b + x \times \sqrt{a^2 - x^2}}{x}$.

Fig. 2. 2. In the inferior conchoid $IE = \sqrt{a^2 - x^2}$, for the same reasons as above; and $PE = b - x$. Proceeding there-

therefore as in the last article, the equation of the inferior conchoid is $y = \frac{b-x \times \sqrt{a^2-x^2}}{x}$. S E C T.
I.

3. In the nodated conchoid $IE = \sqrt{a^2-x^2}$, as in the other two, and $PE = b-x$; and therefore the equation of the nodated conchoid is also $y = \frac{b-x \times \sqrt{a^2-x^2}}{x}$. Fig. 3.

4. The foregoing equations being cleared of the surd, the equation of the superior conchoid becomes $x^4 + 2bx^3 + b^2x^2 - a^2x^2 + y^2x^2 - 2a^2bx = a^2b^2$; and the equation of each of the other two becomes $x^4 - 2bx^3 + b^2x^2 - a^2x^2 + y^2x^2 + 2a^2bx = a^2b^2$.

SCHOLIUM.

Nicomedes, the inventor of the conchoid, published an account of an instrument for the description of the curve*, constructed upon the principles stated in the first Definition, of which the following is the substance.

Let TX , PR be two rulers of wood or metal, fixed at right angles to one another at R ; and let them be of indefinite length, and have each a smooth groove to a convenient extent, as represented. Let DL be another ruler of wood or metal, of indefinite length, and let it also have a smooth groove of a convenient extent, as represented. Let P be a pin which may be fixed in the ruler PR at any requisite distance from the point R . Let A be a pin which may be fixed in the ruler DL at any requisite distance from the extremity D . Let the ruler DL be adapted to the other two by means of the pins A , P in such a manner, that the pin A , fixed in DL , may slide smoothly in the groove in TX , and that the groove in DL may always embrace,

Fig. 4.

* See the Oxford edition of Archimedes, page 147.

S E C T. and the fix first Definitions, let $D E$ be an ordinate to the axis according to the seventh Definition, and let it meet the axis in E ; and then $R E$ is to $R V$ as $P E$ to $P D$.

For $B E$ is a parallelogram, and therefore (34. i.) $R E$ is equal to $B D$, and, by the Cor. to the Definitions, $A D$ is equal to $R V$. Consequently $R E : R V :: B D : A D$. But the triangles (29. i.) $B D A$, $E P D$ are equiangular, and therefore (4. vi.) $B D : A D :: P E : P D$. Consequently (11. v.) $R E : R V :: P E : P D$.

Cor. 1. The rest remaining as above, with R as a center and $R V$ as a distance describe $V I C$ a quadrant of a circle, and let it cut $D E$, or $D E$ produced, in I , and draw $I R$. Then $I R$ is equal to $R V$, and therefore by the above $R E : I R :: P E : P D$. In the superior conchoid the angle at E is common to the two triangles $R E I$, $P E D$, and in the other two conchoids, the angle at E in the triangle $R E I$ is equal to the angle at E in the triangle $P E D$, each of them being a right angle. Consequently (7. vi.) the triangles $R E I$, $P E D$ are equiangular, and therefore (4. vi.) $P E : E D :: R E : I E$.

Cor. 2. From the preceding Cor. the equation of each of the conchoids may be easily deduced. For in each of them put $R V = a$, $P R = b$, $R E = x$, and $D E = y$.

Fig. 1. 1. In the superior conchoid (47. i.) $I E = \sqrt{a^2 - x^2}$, as $I R$, $R V$ are equal; and $P E = b + x$. Consequently $b + x : y :: x : \frac{xy}{b + x} = I E = \sqrt{a^2 - x^2}$; and $y = \frac{b + x \times \sqrt{a^2 - x^2}}{x}$.

Fig. 2. 2. In the inferior conchoid $I E = \sqrt{a^2 - x^2}$, for the same reasons as above; and $P E = b - x$. Proceeding there-

therefore as in the last article, the equation of the inferior conchoid is $y = \frac{b-x \times \sqrt{a^2-x^2}}{x}$. SECT.
I.

3. In the nodated conchoid $IE = \sqrt{a^2-x^2}$, as in the other two, and $PE = b-x$; and therefore the equation of the nodated conchoid is also $y = \frac{b-x \times \sqrt{a^2-x^2}}{x}$. Fig. 3.

4. The foregoing equations being cleared of the surd, the equation of the superior conchoid becomes $x^4 + 2bx^3 + b^2x^2 - a^2x^2 + y^2x^2 - 2a^2bx = a^2b^2$; and the equation of each of the other two becomes $x^4 - 2bx^3 + b^2x^2 - a^2x^2 + y^2x^2 + 2a^2bx = a^2b^2$.

SCHOLIUM.

Nicomedes, the inventor of the conchoid, published an account of an instrument for the description of the curve*, constructed upon the principles stated in the first Definition, of which the following is the substance.

Let TX, PR be two rulers of wood or metal, fixed at right angles to one another at R ; and let them be of indefinite length, and have each a smooth groove to a convenient extent, as represented. Let DL be another ruler of wood or metal, of indefinite length, and let it also have a smooth groove of a convenient extent, as represented. Let P be a pin which may be fixed in the ruler PR at any requisite distance from the point R . Let A be a pin which may be fixed in the ruler DL at any requisite distance from the extremity D . Let the ruler DL be adapted to the other two by means of the pins A, P in such a manner, that the pin A , fixed in DL , may slide smoothly in the groove in TX , and that the groove in DL may always embrace,

Fig. 4.

* See the Oxford edition of Archimedes, page 147.

SECT. I. but slide smoothly over the pin P, fixed in P R. Then if a pencil or pen be attached to the fixed point D in D L, it will trace out the superior, inferior, or nodated conchoid, according as the conditions stated in the fourth, fifth, or sixth Definition, are attended to in adjusting the instrument.

PROP. III. PROB. I.

Two straight lines being given, let it be required to find two means in continued proportion between them by a conchoid.

Fig. 5. Let L G, L A be two given straight lines ; it is required to find two means in continued proportion between them.

Let A L, L G be at right angles to one another, and complete the parallelogram A L G B. Bisect A B in D, and B G in E. Draw L D, and, being produced, let it meet G B produced in H. Draw E C perpendicular to B G, and let it meet G C equal to A D, or D B, in C. Draw H C, and G F parallel to it. With C as a pole, G F as a directrix, and a distance equal to A D or G C between the directrix and vertex, let a conchoid be described, and let it cut H G K in K ; and F K will be equal to A D or G C, by the Cor. to the Definitions. Draw K L, and, being produced, let it meet B A in M. Then will G K, M A be the two mean proportionals required.

For (6. ii.) the rectangle under B K, K G, together with the square of E G, is equal to the square of E K ; and therefore, by adding the square of E C to these equals, the rectangle under B K, K G, together with the square of C G, (47. i.) is equal to the square of C K. By similar triangles M A : A B :: M L : L K :: B G : G K ; and therefore (16. vi.) M A \times G K is equal to A B \times B G.

$\times B G$. Again, by similar triangles, $L G$ or $A B : D B :: S E C T. I.$
 $G H : B H$, and as $A B$ is double of $D B$, $H G$ is double
of $B G$, and therefore $A D \times H G$ is equal to $A B \times B G$.
Consequently (16. vi.) $M A : A D :: H G : G K :: C F :$
 $F K$; and (18. v.) $M D : A D :: C K : F K$. But $A D$
is equal to $F K$, by construction, and therefore (14. v.)
 $M D$ is equal to $C K$, and $M D^2$ is equal to $C K^2$. By
the above therefore the rectangle under $B K$, $K G$, to-
gether with the square of $C G$, is equal to the square
of $M D$, which (6. ii.) is equal to the rectangle under
 $B M$, $M A$ together with the square of $A D$, or its equal
 $C G$. Consequently $B K \times K G$ is equal to $B M \times M A$,
and (16. vi.) $B M : B K :: G K : M A$. But by similar
triangles $B M : B K :: G L : G K :: M A : A L$. Con-
sequently (11. v.) $G L : G K :: G K : M A :: M A :$
 $A L$; and therefore $G K$, $M A$ are two means in con-
tinued proportion between the given straight lines $A L$,
 $L G$.

PROP. IV. PROB. II.

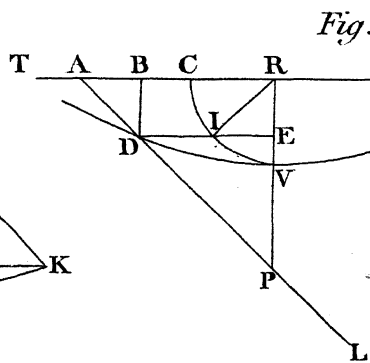
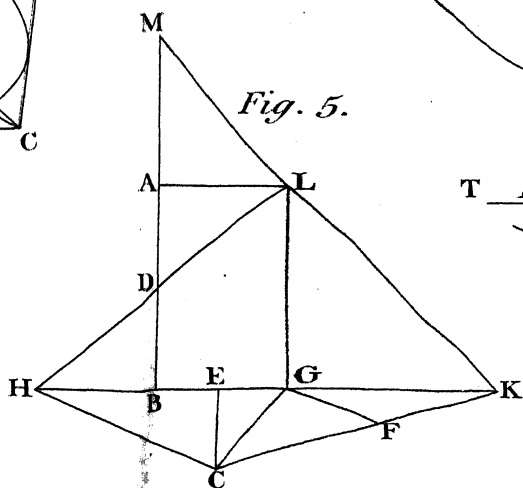
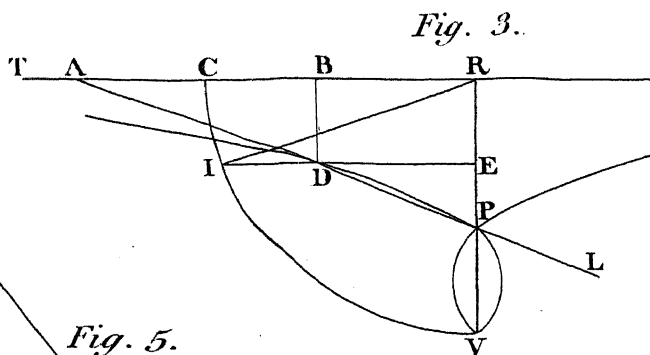
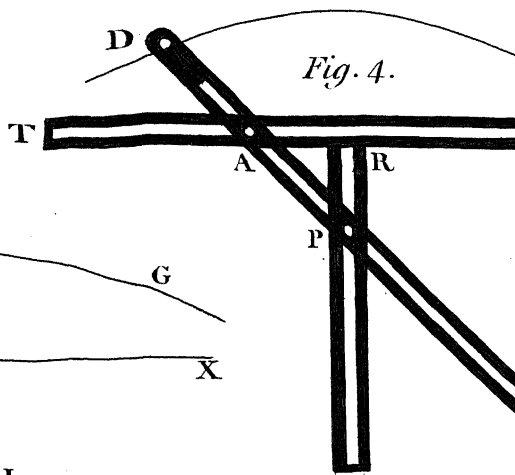
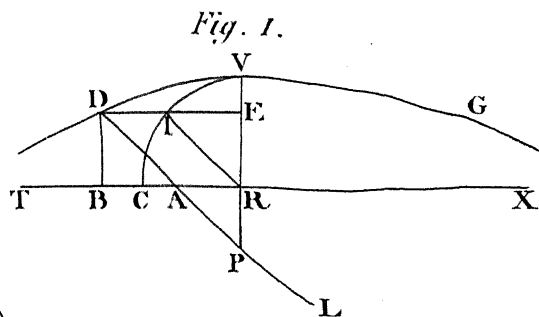
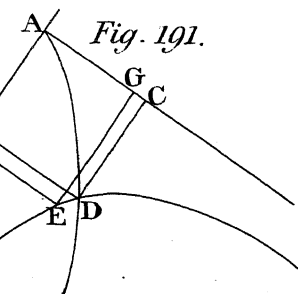
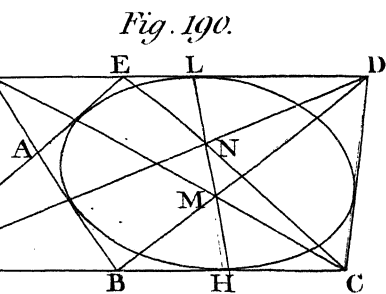
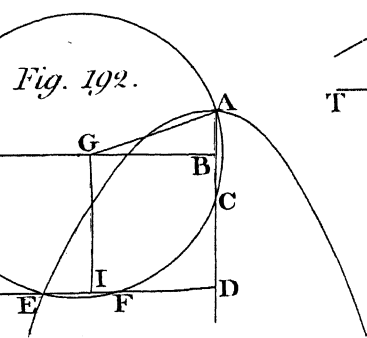
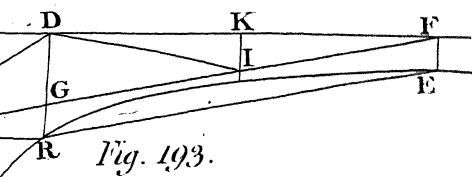
An angle being given, let it be required to divide it into three equal parts, and by means of a conchoid.

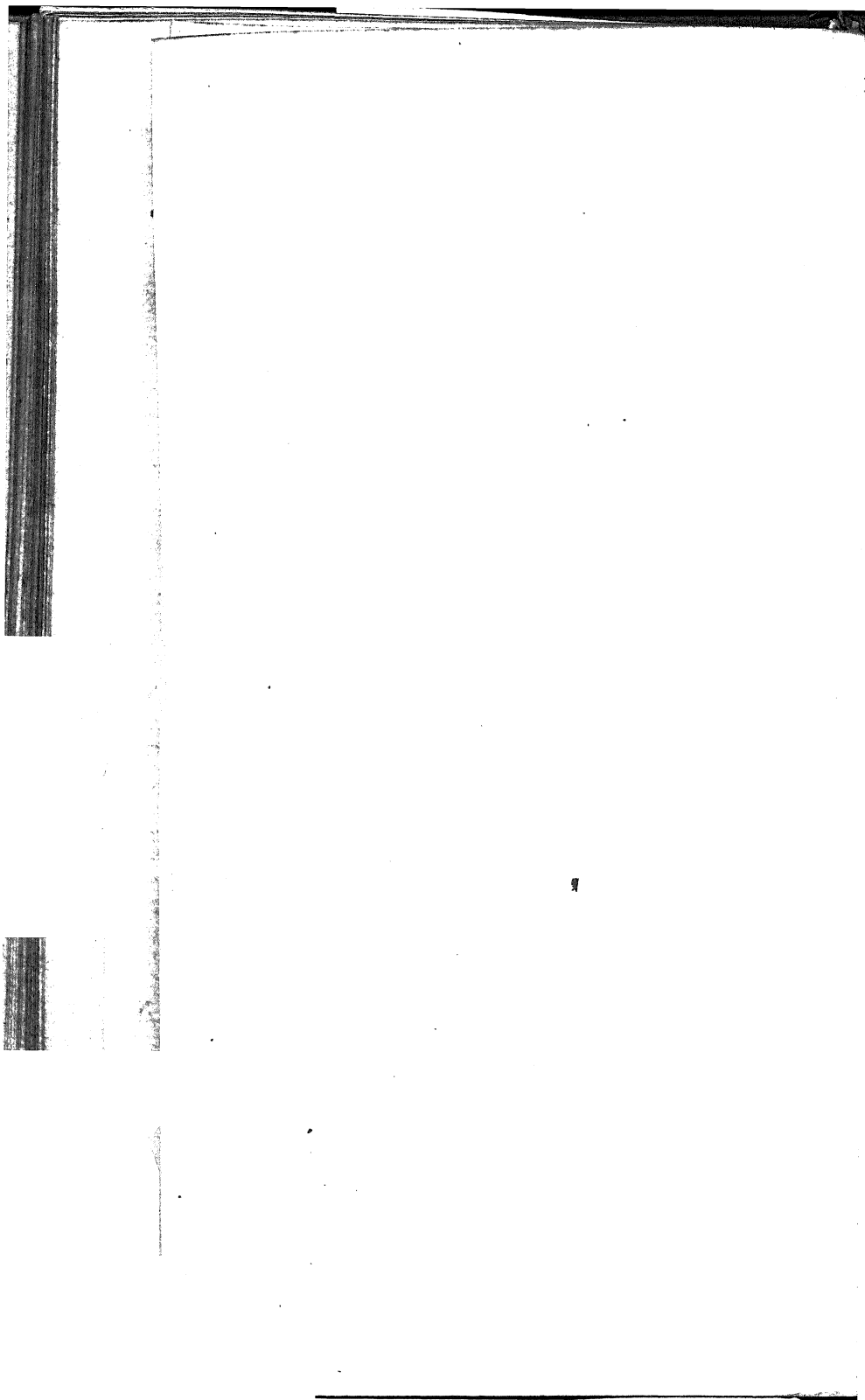
Let $A C B$ be a given angle, it is required to trisect it, or divide it into three equal parts. Fig. 6.
7.

With c as a center, and any convenient distance $c A$, describe the circle $A B F$, and produce the diameter $B F$ indefinitely. With A as a pole, $B F$ as a directrix, and a distance between the directrix and vertex equal to $c A$, let a conchoid be described, and let it cut the circumference of the circle in E . Draw $A E$, and let it meet $B F$ in D , and the angle $A D C$ is a third part of the given angle $A C B$.

For $C E$ being drawn, $E D C$ is an isosceles triangle, and (5. i.) the angles $E D C$, $E C D$ are equal, as are

SECT. I. also the angles CEA , CAE . In Fig. 6. the angle AEC (32. i.) therefore is double of the angle ADC , and consequently the angle CAD is double of the angle ADC . But the angle ACB (32. i.) is equal to the angles CAD , ADC together, and therefore the angle ADC is a third part of the angle ACB . In Fig. 7. the angle EAC , (32. i.) and therefore its equal AEC , is equal to the angles EDC , ACD together; and the angle DAC is equal to the angles AEC , ACE together. Consequently the angle DAC is equal to the angles EDC , ACD , ACE taken together; that is, the angle DAC is equal to the angles EDC , ECD together, or to the double of the angle ADC . Consequently, as the angle ACB (32. i.) is equal to the angles DAC , ADC together, the angle ADC is a third part of the angle ACB .





SECTION II.

Of the Cissoid.

LEMMA.

Let $EGFH$ be a circle of which c is the center, and EF, HG two diameters at right angles to one another. Let EF be produced to P , so that FP may be equal to CF ; and A being any point in CG draw AP , and upon AP as a diameter describe the semicircle ABP . Then if from A a straight line AB , equal to CP , be inscribed in the semicircle, the point in which AB is bisected will be on the same side of CP with the point A .

Fig. 3.

For let K be the point in which AP is bisected, and draw KF ; and draw KD perpendicular to AB . Then as ACP is a right angle, the circumference ABP (31. iii.) passes through c ; and as AB, CP are equal, the circumference ACB (28. iii.) is equal to the circumference PBC ; and therefore the circumference AC is equal to the circumference BP . The angle APC (27. iii.) is therefore equal to the angle PAB ; that is, the angle KPF is equal to the angle KAD . Again, as PK is equal to KA , and PF equal to FC , $PK : KA :: PF : FC$; and therefore (2. vi.) KF is parallel to CA , and (29. i.) the angle PFK is a right angle. The triangles PKF, AKD are therefore equiangular, and (26. i.) KF is equal to KD , and AD to PF . Consequently if with K as a center and KF as a distance a circle be described, it will pass through D , and CP will be a tangent to this circle (16. iii.) as it is at right angles to KF . The straight line AB is also bisected in D ,
for

SECT. II. for AD is equal to PF . Consequently the point in which AB is bisected is on the same side of CP with the point A .

DEFINITIONS.

I.

Fig. 9. Let $EGFH$ be a circle, of which C is the center, and EF , HG two diameters at right angles to one another. Let EF be produced to P , so that FP may be equal to CF the radius, and let HG be produced indefinitely on one side towards A a point in HG . Let the straight line BL , of indefinite length towards L , pass through P ; and let BA , equal to EF or CP , be at right angles to BL , and let D be the point in which BA is bisected. Then if the straight lines BL , BA be so moved that BL always pass through P , and the extremity A of BA be always in HG , the point D will describe a curve FGD which is called a *Cissoid*, or the *Cissoid of Diocles*.

II.

The circle $EGFH$ is called the *Generating Circle* of the cissoid; and the point P is called its *Pole*.

III.

The straight line HGA is called the *Directrix* of the cissoid.

IV.

The diameter EF of the generating circle is called the *Axis* of the cissoid, and the point F is called its *Cusp*.

V.

The straight line EK perpendicular to the axis is called the *Asymptote* to the cissoid.

VI.

A straight line drawn from any point in the cissoid perpendicular to the axis is called an *Ordinate* to the axis,

axis, and the segment of the axis between the cusp and an ordinate is called an *Absciss* of the axis. S E C T.
II.

PROP. I.

The cissoid commences at the cusp, and the curve is entirely on one side of the axis; it also passes through the point in which the directrix cuts the generating circle, and being continually produced it approaches nearer and nearer to the asymptote, but never meets it.

For, the rest remaining as in the Definitions, draw the straight line PA . Then as PCA , PBA are right angles, a semicircle described upon PA as a diameter will (31. iii.) pass through B and C ; and the straight line AB , equal to CP , will be inscribed in this semicircle, in every situation of A , regulated according to the first Definition. Consequently when A coincides with c the straight line AB will coincide with CP , and the point D will coincide with F . If therefore the description of the curve be supposed to begin from this situation, the cusp F will be the point at which it commences, and as soon as A has moved from c towards G the describing point D will be removed from EF towards G , according to the Lemma prefixed to the Definitions. When the distance of A , in the directrix, from c is equal to CP or AB , then the point B will coincide with c , and the describing point D will coincide with G . Through D draw KM perpendicular to CA , and let it meet the asymptote in K . Let BP cut CA in N . Then as PCN , NBA are right angles, and (15. i.) as the angles PNC , ANB are equal, the triangles PCN , ABN are equiangular. The triangles PCN , AMD are therefore equiangular, and the angle CPN is equal to the angle MAD . But as the point A recedes from G the point N also recedes from it, and there-

Fig. 9.

SECT. II. therefore the cissoid being continually produced the angle PNC becomes less and less, and consequently the angle CPN , or its equal MAD , becomes greater and greater. The perpendicular DM must therefore continually increase, as the length of DA is constant, and consequently DK , the distance of the curve from the asymptote, must continually decrease, as KM (34.i.) is equal to EC . The point D however can never fall into EK , as DM can never become equal to DA or EC ; for if it did then PB would become parallel to CA ; which is impossible.

Cor. From the above it is evident, that an ordinate drawn from any point in the cissoid to the axis will cut the generating circle.

PROP. II.

An ordinate, drawn from any point in the cissoid to the axis, is a third proportional to its segment between the generating circle and the axis, and the corresponding absciss.

Fig. 10.
11.

From D , any point in the cissoid $F D G$ whose cusp is F , let DR be drawn an ordinate to the axis FE , and let it cut the generating circle in T ; then TR is to RF as RF to RD .

For, the rest remaining as in the Definitions and preceding Proposition, let DR cut the generating circle again in S , and draw CS . Then as AD is equal to CS , and (34. i.) DM equal to RC , and DMA , CRS right angles, the squares of MA , SR are (47. i.) equal; and consequently MA , SR are equal. Again, as in the triangles ABN , PCN , AB , PC are equal, and as the angle BNA is equal to the angle CNP , and the angle ABN equal to the angle PCN , the side BN is equal to the side CN . By similar triangles also $DA : MA ::$

NA

SECT.
II.

$NA : BA$, and $MD : MA :: NB : BA$. Consequently on account of the equals (24. and its first Cor. v.) $FR : SR :: CA : EF$; and $FR \times EF$ is equal to $SR \times CA$. But (3. ii.) $FR \times EF$ is equal to $ER \times RF$ together with the square of FR ; and (35. iii.) $ER \times RF$ is equal to the square of SR . Consequently $FR \times EF$ is equal to the sum of the squares of FR , SR . Again, $SR \times CA$ is equal to $SR \times SD$, for (34. i.) RD , CM are equal, and by the above MA is equal to SR . But $SR \times SD$ (3. iii.) is equal to $SR \times RD$ together with the square of SR . By the above therefore the sum of the squares of FR , SR is equal to $SR \times RD$ together with the square of SR . Consequently $SR \times RD$ is equal to the square of FR ; and therefore (3. iii.) as TR is equal to SR , (17. vi.) $TR : RF :: RF : RD$.

Cor. 1. The straight lines ER , RT , RF , RD are in continued proportion. For (Cor. 8. vi.) $ER : RT :: RT : RF$; and as above $RT : RF :: RF : RD$.

Cor. 2. The equation of the cissoid is easily deduced from the last Corollary. For put $EF = a$, $FR = x$, and $RD = y$; and then $ER = a - x$, and $RT^2 = ER \times RF$, or $RT = \sqrt{ax - x^2}$. Consequently $\sqrt{ax - x^2}$

$: x :: x : \frac{x^2}{\sqrt{ax - x^2}} = y$, and $\frac{x^4}{ax - x^2} = y^2$, or $\frac{x^3}{a - x} = y^2$;

and therefor $x^3 = ay^2 - xy^2$, which is the equation of the curve.

PROP. III.

If from the cusp of a cissoid a straight line be drawn cutting the cissoid and the generating circle, straight lines drawn through the points of section, and perpendicular to the axis, will be equally distant from the directrix.

Let $FDGL$ be a cissoid, of which F is the cusp, FE the axis, HCG the directrix, and $EGFH$ the generating

Fig. 12.

SECT. II. ing circle. From F draw any straight line FK cutting the cissoid in D and the circle in K , and let DB, KM be perpendicular to the axis; the straight lines DB, KM are equally distant from HCG .

For let c be the center of the circle, and let the directrix cut the circle and cissoid in G . Let BD meet the circle in A , and let KM meet it again in N , and draw EK . Then, by Prop. II. $AB : BF :: BF : BD$; and (4. vi.) $BF : BD :: MF : MK$. Consequently (II. v.) $AB : BF :: MF : MK$; and therefore (6. vi.) the triangles AFB, FKM are equiangular, and the angle AFB is equal to the angle MKF . The circumference AE therefore (26. iii.) is equal to the circumference NF , which is equal to the circumference FK . The circumference FA is therefore equal to the circumference EK , and consequently (29. iii.) the straight lines FA, EK are equal. But (Cor. 8. vi.) EM is a third proportional to EF, EK ; and FB is a third proportional to EF, AF . Consequently EM, FB are equal, and therefore CM is equal to CB .

Cor. 1. Hence it is evident that the arch GK is equal to the arch GA .

Cor. 2. If equal arches as GK, GA be set off in the circumference of the circle, on the opposite sides of HCG , and perpendiculars KM, AB be drawn to the diameter EF , a straight line drawn from F to K will cut the perpendicular AB and the cissoid in the same point. For the same reasons a straight line drawn from F to A will cut the perpendicular KM and the cissoid in the same point. For it may be proved, as above, if the straight line FA cut the generating circle in A and the cissoid in L , and if AB, LM be drawn perpendicular to the axis, that CB, CM are equal.

SCHOLIUM.

SECT.
II.

Diocles, the inventor, considered the property expressed in the last Corollary as the primary one of the cissoid; and he supposed the description of the curve to be effected by means of an indefinite number of points, as D and L , obtained from equal arches as GK , GA , and the intersections of the straight lines FK , FA with the perpendiculars AB , ML *. Sir I. Newton first shewed how the cissoid might be described by continued motion according to the conditions expressed in the first Definition in this section †. The method will be easily understood if PCA , LBA be supposed to be two squares of wood or metal, P being a fixed point in PC one arm of the one, and A a fixed point in BA an arm of the other; and if the adjustment of the instrument and its action be supposed to be regulated as mentioned in the first Definition.

Fig. 9.

It is evident from the Definitions, and the Propositions and their Corollaries, that with the same generating circle $EGFH$, the same pole P , and the same cusp F , another cissoid FHR may be described on the opposite side of EF to that on which FGD is described. It is also evident that SEK , touching the generating circle in E , is the common asymptote to these two cissoids, and that EF is their common axis; and it may readily be perceived that the properties proved of the one equally apply to the other.

* See the Oxford edition of Archimedes, page 138.

† See the Appendix to the Arithmetica Universalis.

SECTION III.

Of the Quadratrix.

DEFINITIONS.

I.

Fig. 13. Let ADE be a femicircle, and let c be the center of the circle, and AE a diameter. Let CD be at right angles to AE and meet the circumference in D , and let it be produced to s so that DS may be equal to CD . Let a straight line CH of indefinite length revolve about c , begin its revolution from a coincidence with CE , and move towards D ; and at the same time that CH begins to revolve let a straight line FG move from a coincidence with CE towards s , and let FG be always parallel to AE , or perpendicular to CS . Let CH revolve and FG move with an uniform velocity, and let the arch EH passed over by the revolution of CH be to the distance CF moved over by the extremity of FG in the same time, as the quadrantal arch ED to the radius CE ; the curve BID described by the point I , in which CH , FG cut one another, is called a *Quadratrix*, or the *Quadratrix of Dinostrates*.

II.

The circle ADE is called the *Generating Circle* of the quadratrix.

III.

The straight line CS is called the *Axis* of the quadratrix.

IV.

If B be the point at which the lines CH , FG commence

mence their interfection, the straight line CB is called SECT.
III.
the *Base* of the quadratrix.

V.

The straight line ST perpendicular to the axis is called the *Asymptote* to the quadratrix.

PROP. I.

The curve of the quadratrix passes through the point in which the axis cuts the generating circle; and being continually produced it approaches nearer and nearer to the asymptote, but never meets it.

Part I. Every thing remaining as in the Definitions, Fig. 13.
as CH revolves and FG moves with an uniform velocity, and as the velocity with which CH revolves is to the velocity with which FG moves as the quadrantal arch ED to the radius CD , the arch EH is to CF as the quadrantal arch ED to the radius CD . When the revolving line CH therefore coincides with CD , the point F in FG will coincide with D , and I will also coincide with D . Consequently the curve of the quadratrix must pass through D .

Part II. The rest remaining as above, let CL represent the revolving line after it has proceeded beyond CD from E , and let LM represent the situation of the line moving parallel to AE at the same time, and let CL , ML intersect one another in L , and let CL cut the generating circle in K . Then it is evident, from the first Definition, that the point L is in the curve of the quadratrix; and, for the same reasons as above, the arch EDK is to CM as the quadrantal arch ED to the radius CD . Hence it is evident, that, if the curve of the quadratrix be continually produced, the revolving line will cut off greater and greater arches of the generating circle, reckoning from the extremity E , and

SECT.
III.

consequently the distances of the moving line parallel to AE must become greater and greater. The distance of the moving line parallel to AE from the asymptote TS must therefore continually decrease. Consequently the curve of the quadratrix must approach nearer and nearer to the asymptote upon being continually produced, but they can never meet, for if they did then the revolving line would coincide with AE , and AE , TS would meet. But this is impossible, for, by the fifth and first Definitions, (and 28. i.) they are parallel.

Cor. Any arch EH and distance CF , passed over by the revolving line CH and moving line FG in the same time, are to one another as the quadrantal arch ED to the radius CD . Also (19. v.) the arch DH is to DF as the quadrantal arch ED to the radius CD .

PROP. II.

The base of the quadratrix is a third proportional to the quadrantal arch of the generating circle and its radius.

Fig. 14. Let BID be a quadratrix, of which CE is the base, ADE the generating circle, and CE or CD the radius of the circle, c being its center; the quadrantal arch EHD of the generating circle is to its radius CE as CE to CB .

For let CH be a position of the revolving line, and FI a corresponding position of the line which moves parallel to AE , and let them intersect one another in I , as in the first Definition, so that I may be in the curve of the quadratrix. Let CH , FI be indefinitely near to CE , so that the arch EH of the generating circle may be indefinitely small; and let HL , IK be perpendicular to CE . Then, by the *Cor.* to Prop. I. the arch EH : CF :: arch EHD : CE . But it is evident that FK is a parallelogram, and therefore (34. i.) CF is equal to

KI ;

$\kappa \iota$; and as the arch $\epsilon \eta$ is indefinitely small, it is equal to its sine $\iota \eta$. Consequently $\iota \eta : \kappa \iota :: \text{arch } \epsilon \eta : \text{ch } \epsilon \eta$; and therefore (II. v. and 4. vi.) $\text{arch } \epsilon \eta : \text{ch } \epsilon \eta :: \text{cl} : \text{ck}$. But when the arch $\epsilon \eta$ is indefinitely small, and equal to $\iota \eta$, cl is equal to the radius, and ck becomes equal to cb . Consequently the quadrantal arch $\epsilon \delta : \text{ch } \epsilon \delta :: \text{ce} : \text{cb}$.

SECT.
III.

Cor. 1. As by the above and inversion $\text{cb} : \text{ce} :: \text{ce} : \text{the quadrantal arch } \epsilon \delta$, by the *Cor.* to Prop. I. (and II. v.) $\text{cb} : \text{cd} :: \text{fd} : \text{the arch } \delta \eta$. Also by the above, *Cor.* to Prop. I. (and II. v.) $\text{cb} : \text{cd} :: \text{cf} : \text{the arch } \epsilon \eta$.

Cor. 2. The equation of the quadratrix may be obtained from the above in the following manner. Put $\text{cd} = a$, $\text{cb} = b$, $\epsilon \eta = z$, $\text{cf} = y$; and then, as in the Proposition $b : a :: a : \frac{a^2}{b} = \text{the quadrantal arch } \epsilon \delta$. Consequently $\frac{a^2}{b} : a :: z : \frac{bz}{a} = \text{cf} = y$; and $bz = ay$, the equation of the curve.

LEMMA.

A circle is equal to a right angled triangle, which has one of the sides round the right angle equal to the radius of the circle, and the other side equal to the circumference.

Let AIBDE be a circle, as in Fig. 15. of which c is the center, and ci a radius, and let kmn , as in Fig. 16. be a right angled triangle, having km one of the sides round the right angle at m equal to ci , and the other mn equal to the circumference of the circle; the circle AIBDE is equal to the triangle kmn .

For, if it be possible, let the circle be greater than the triangle, and first by inscribing a square in it, and afterwards by a repeated bisection of circular arches,

SECT. III. let a polygon of an even number of sides be inscribed in the circle; and let the excess of the circle above the polygon be less than its excess above the triangle. Then the polygon thus inscribed in the circle will be greater than the triangle. Let IB be a side of this polygon, and let CL , at right angles to it, meet it in I . Then CL is less than KM , and as the straight line IB is less than the circular arch IB , the perimeter of the polygon is less than the circumference of the circle. The rectangle under CL and the perimeter of the polygon is therefore less than the rectangle under KM , MN . But it is evident (from 1. ii. and 34. i.) that the rectangle under CL and the perimeter of the polygon is double the area of the polygon; and the rectangle under KM , MN is double the area of the triangle KMN . The polygon is therefore less than the triangle KMN ; and it is also greater; which is absurd.

But, if it be possible, let the circle be less than the triangle; and first by describing a square about the circle, and afterwards by a repeated bisection of circular arches let a polygon be described about the circle, and let the excess of the polygon above the circle be less than the excess of the triangle above the circle. And that this may be done is evident from (1. x. and 1. xii.) considering that if BH , AH touch the circle in B and A and meet one another in H , then if CH be drawn cutting the circle in I , and FG touch it in I and meet BH in F and AH in G , and BI , IA be drawn, the triangle HIF is greater than the triangle BIF , and the triangle HIG is greater than the triangle AIG . For (18. iii.) HIF , HIG are right angles, and therefore (18. i.) FH is greater than FI , and GH greater than GI . But it is evident (from 36. iii.) that BF is equal to FI , and GI to GA , and therefore (1. vi.) the triangle HIF is greater than the triangle BIF , and the tri-

SECT.
III.

triangle HIG is greater than the triangle AIG . Let FG be a side of the polygon described about the circle, whose excess above the circle is less than the excess of the triangle KMN above the circle, and then the polygon, of which FG is a side, is less than the triangle KMN . But as the perimeter of the circumscribed polygon is greater than the circumference of the circle, and as the rectangle under CI and the perimeter of the circumscribed polygon is equal to the double of the area of the polygon, it is evident for the same reasons as above that the circumscribed polygon is greater than the triangle KMN . The circumscribed polygon therefore is both less and greater than the triangle KMN ; which is absurd. Consequently the circle $AIBDE$ is equal to the triangle KMN .

Cor. 1. A circle is equal to a rectangle which has one of its sides equal to the radius of the circle, and the other side round the same angle equal to half the circumference.

Cor. 2. The diameter of one circle is to the diameter of another as the circumference of the first mentioned to the circumference of the other. For put D equal to the diameter of the one and c equal to its circumference, and put d equal to the diameter of the other and c equal to its circumference. Then, by the preceding *Cor.* (and 2. xii.) $D \times c : d \times c :: D^2 : d^2$, and $D \times c : D^2 :: d \times c : d^2$. Consequently (1. vi.) $c : D :: c : d$.

Cor. 3. If the two circles $PHGB$, $KLME$ have the common center C , and CA , CB be drawn cutting the outer circle in A , B , and the inner in D , E , the radius CB is to the radius CE as the arch AB to the arch DE . For let FG , HB be two diameters of the outer circle at right angles to one another, and let FG cut the inner circle in K , M , and HB cut it in L , E . Then

Fig. 17.

S.E.C.T.
III.

(33. vi.) the arch FB : the arch AB :: the angle FCB : the angle ACB :: the arch KE : the arch DE ; and therefore, by alternation, the arch FB : the arch KE :: the arch AB : the arch DE . But the arch FB is a fourth part of the circumference $FHGB$, and the arch KE is a fourth part of the circumference $KLME$. Consequently, by the last Cor. (and 15. v.) $CB : CE ::$ the arch AB : the arch DE .

Such sectors as ACB , DCE which have the angles at their centers equal, are called *Similar Sectors*.

PROP. III.

If from any point in the quadratrix a straight line be drawn to the center of the generating circle, and also a straight line perpendicular to the axis, and if with the center of the generating circle, and the base as a distance, a circle be described, the arch of this circle intercepted between the extremity of the base and the straight line drawn from the quadratrix to the center, will be equal to the segment of the axis between the center and perpendicular.

Fig. 19.

From any point L in the quadratrix BDL let a straight line LC be drawn to C the center of the generating circle ADE , and with C as a center and CB , the base of the quadratrix as a distance, let a circle be described; the arch BF of this circle, intercepted between B and C , is equal to CM , the segment of the axis CDM between C the center and LM the perpendicular.

For, by Prop. II. and inversion, $CB : CD :: CD$: the quadrantal arch DE ; and, by Cor. 1. to Prop. II. CD : quadrantal arch $DE :: CM$: the arch EK ; and, by Cor. 2. to the preceding Lemma, (and 15. v.) $CB : CD$ or $CE ::$ the quadrantal arch BG : the quadrantal arch

OF THE QUADRATRIX.

rch DE. Again (33. vi.) the angle ECD : the angle
 CK :: the quadrantal arch ED : the arch EK ; and
 or the same reasons the quadrantal arch BG is to the
 arch BF in the same proportion. Consequently (11. v.)
 the quadrantal arch ED : the arch EK :: the qua-
 drantal arch BG : the arch BF ; and, by alternation,
 the quadrantal arch ED : the quadrantal arch BG ::
 the arch EK : the arch BF. By the above therefore
 and 11. v.) the arch BF : the arch EK :: CM : the
 arch EK ; and (14. v.) consequently the arch BF is
 equal to CM.

Cor. The quadrantal arch BG is equal to the radius
 CD. For, as above, by Prop. II. CB : CD :: CD :
 the quadrantal arch DE ; and, by Cor. 2. to the pre-
 ceding Lemma, CB : CD :: the quadrantal arch BG :
 the quadrantal arch ED. Consequently (11. v.) CD :
 the quadrantal arch ED :: the quadrantal arch BG :
 the quadrantal arch ED ; and therefore (14. v.) CD is
 equal to the quadrantal arch BG.

SCHOLIUM.

If the straight line CH revolve, and the straight line
 FG move with the same relative velocities as stated in
 the first Definition, but on the side of AE opposite to
 that before supposed, the quadratrix LDB may be ex-
 tended as represented by LDBKO in Fig. 18. and if
 the generating circle be completed, and DC be pro-
 duced to M so that CM be equal to CS, then a straight
 line MN at right angles to CM will also be an asymp-
 tote to the curve. It is also evident that the curve
 will pass through the point K in which CM cuts the
 generating circle.

It is much to be wished that an instrument were de-
 vised for describing the quadratrix by continued mo-
 tion, as the two following important Problems could

Fig. 18.

SECT. III. then be solved geometrically. In the following solutions the possibility of describing the quadratrix is necessarily supposed.

PROP. IV. PROB. I.

From a given rectilineal angle to cut off any part required, by means of the quadratrix.

Fig. 13. Let $\kappa c e$ be a given rectilineal angle; it is required to cut off any part from it, by means of the quadratrix.

With c as a center, and any convenient distance $c e$, let a circle $A D E$ be described. With the generating circle $A D E$ let the quadratrix $B I D L$ be described, and let it meet $c \kappa$ in L . Let $c D M$ be the axis of the quadratrix, and let $L M$ be perpendicular to it. Let $c m$ be so cut in F (9. vi.) that $c m$ may be to $c F$ as the whole given angle $\kappa c e$ to the part required. Draw $F G$ parallel to $c e$, and let it cut the quadratrix in I . Draw $c I$ and let it meet the circle in H ; and the angle $e c H$ will be the part required.

For, by the Cor. to Prop. I. (and 11. v.) $c m : c F ::$ the arch κe : the arch $H e$; and (33. vi.) the arch κe : the arch $H e ::$ the angle $\kappa c e$: the angle $H c e$.

PROP. V. PROB. II.

To find a square equal to a given circle, by means of the quadratrix.

Fig. 18. Let $A D E \kappa$ be a given circle; it is required to find a square equal to it, by means of the quadratrix.

Let c be the center of the circle, and with $A D E \kappa$ as a generating circle, and its diameter $D \kappa$ as an axis, let the quadratrix $L D B \kappa O$ be described, and let $c B$, in the diameter $A c e$, be its base. Through B draw $F a$ parallel to $D \kappa$, and let it meet

$D F$

DP perpendicular to DK in P , and KQ perpendicular to DK in Q . Draw CP , CQ , and let them meet REV parallel to DK in R and V . Draw RW , VX perpendicular to DK . Then the right angled parallelogram $WRVX$ is equal to the given circle $ADEK$, and a mean proportional between the sides WR , RV is the side of the square required.

SECT.
III.

For with C as a center, and CB as a distance, let the circle GBI be described, and let it meet DK in G and I . Then, by the Cor. to Prop. III. the quadrantal arch BG is equal to CD , and the quadrantal arch BI is equal to CK . Again, by the above construction, DB , WB are parallelograms, and therefore (34. i.) DP is equal to CB , and WR is equal to CE . The triangles CDP , CWR are also equiangular, (29. i.) and therefore (4. vi.) $DP : WR :: CD : CW$. Consequently, by Cor. 2. to the Lemma in this section, (and 15. v.) the quadrantal arch DE is equal to CW , and for the same reasons the quadrantal arch KE is equal to CX ; and therefore WX is equal to half the circumference of the circle $ADEK$. The right angled parallelogram $WRVX$ is therefore, by Cor. 1. to the Lemma in this section, equal to the circle $ADEK$; and a mean proportional (13. vi.) being found, it will be equal to the side of the square required.

Cor. It is evident, from the Cor. to Prop. III. and Cor. 1. to the Lemma in this Section, that the right angled parallelogram $DPQK$ is equal to the circle GBI .

SECTION IV.

Of Cycloids.

DEFINITIONS.

I.

Fig. 20. If the circle AWE roll along the straight line AB so that every part of the circumference may touch it in regular succession, and if at the commencement of the revolution of the circle it touch AB in A , and if during the revolution the point A remain fixed in the circumference, and at the end of it meet the straight line in B ; the curve AVB described by the point A , during the revolution, is called a *Cycloid*.

II.

The circle AWE is called the *Generating Circle* of the cycloid.

III.

The straight line AB is called the *Base* of the cycloid.

Cor. As every part of the circumference of the circle touches the base, in regular succession, the base of the cycloid is equal to the circumference of the generating circle.

IV.

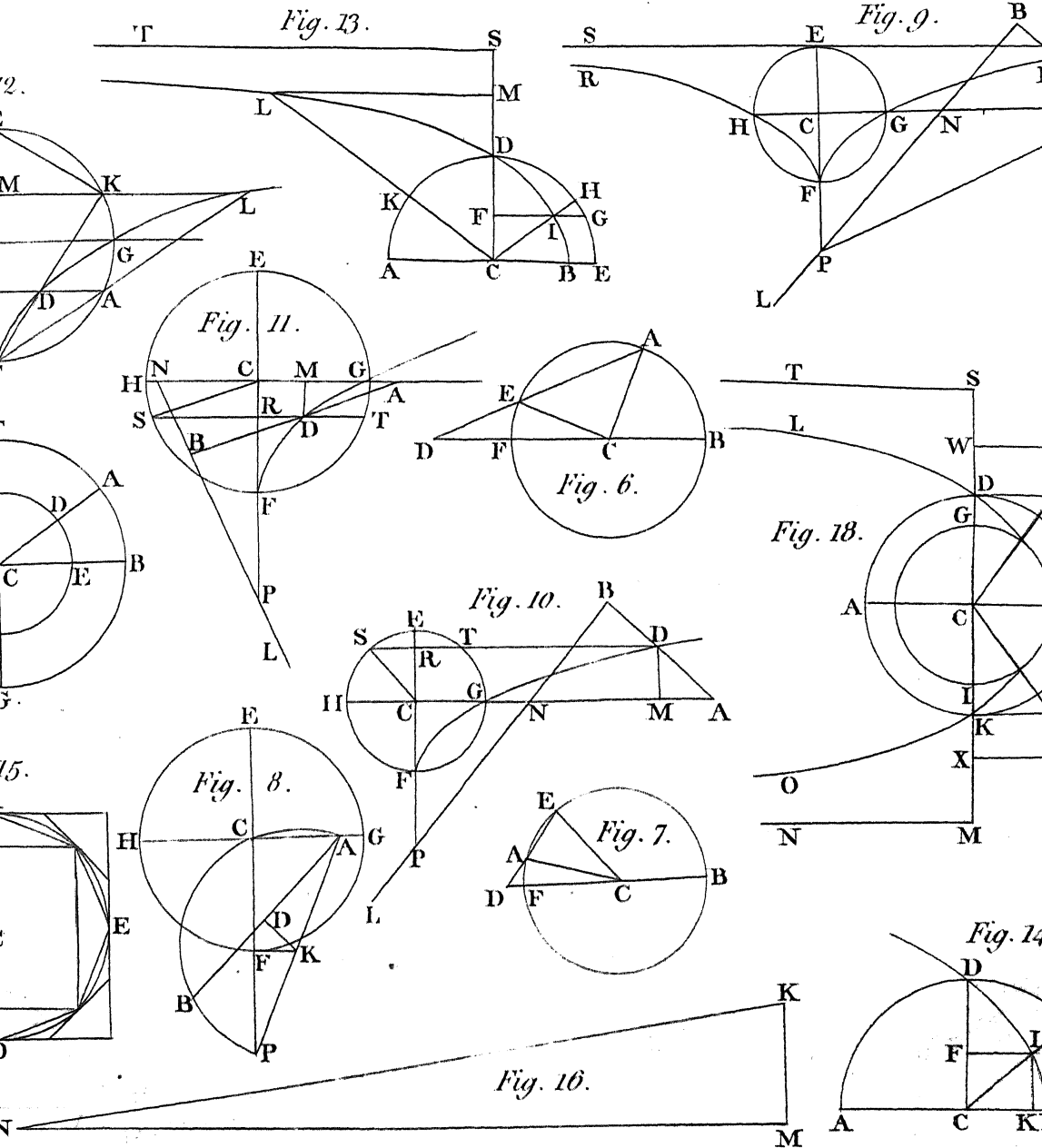
The straight line HV , bisecting the base AB at right angles and meeting the curve in V is called the *Axis* of the cycloid.

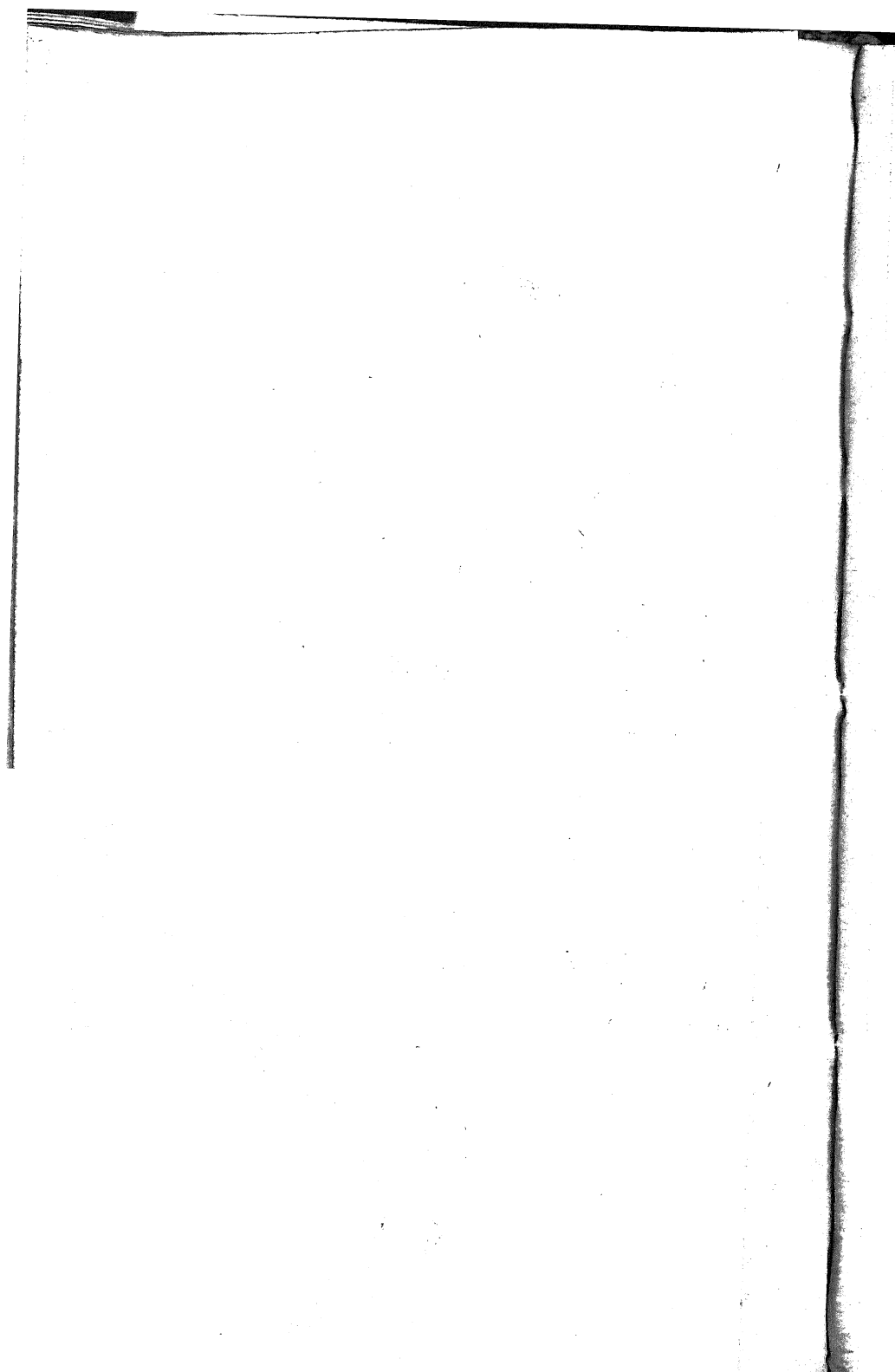
V.

The point V is called the *Vertex* of the cycloid.

VI.

A straight line drawn from any point in the curve per-





perpendicular to the axis is called an *Ordinate* to the axis; and the segment of the axis between the vertex and an ordinate is called an *Absciss*. SECT.
IV.

PROP. I.

During the revolution of the generating circle its center describes a straight line equal to the base of the cycloid; and the axis of the cycloid is equal to the diameter of the generating circle

For, the rest being as in the Definitions, let c be the center of the generating circle, and AE the diameter passing through A . Then as the circle AWH always touches the straight line AB , the center c is at the same distance from AB throughout the revolution, and therefore it must describe a straight line. At the commencement of the revolution, and also at the end of it, AC (18. iii.) is perpendicular to AB , and therefore it is evident that if these perpendiculars and the line described by the center were drawn, a parallelogram would be formed, of which the line described by the center and AB would be opposite sides. The straight line described by the center is therefore equal to the base of the cycloid. Lastly, let AWH be the position of the generating circle at the beginning of the revolution, and then it is evident, from the first five Definitions, that at the middle of the revolution the point H will coincide with H and the describing point A with v . The axis Hv therefore of the cycloid is equal to the diameter of the generating circle.

Cor. A circle described on the axis of a cycloid is equal to the generating circle.

PROP.

SECT.
IV.

PROP. II.

An ordinate drawn from any point in the cycloid to the axis is equal to the arch of the generating circle, described on the axis, between the vertex and ordinate, together with the sine of the same arch.

Fig. 20. From any point F in the cycloid $AVFB$ let FM be drawn an ordinate to HV the axis, and let it cut the circle VPH described on the axis in the point P ; the ordinate FM is equal to the arch VP , between V the vertex and the ordinate, together with PM the sine of the same arch.

For let $SELI$ represent the generating circle when the describing point coincides with F , and in this situation let T denote its center, and let I be the point in which it touches AB . Draw the diameters LTS , FTI , and let LTS meet FM in N . Then SL (18. iii.) is perpendicular to AB ; and, as FM is perpendicular to VH , ML is a parallelogram, and therefore (34. i.) MH is equal to NL , and MN is equal to HL . But, from the Cor. to the third Definition, the semicircle ILF is equal to HB half the base, and from the description of the curve the circular arch LF is equal to the remaining part LB of the base. Consequently HL is equal to the arch IL , or to its equal (26. iii.) SP ; and as HM , LN are equal, it is evident (from Cor. 8. vi.) that MP is equal to NF , and the arch VP equal to the arch SP . The arch VP is therefore equal to MN ; and as NF is equal to PM , the ordinate FM is equal to the arch VP together with PM the sine of the same arch.

Cor. The equation of the cycloid is immediately obtained from the above. For put $FM = y$, the arch $VP = z$, and $PM = s$, and then $z + s = y$, the equation of the curve.

DE-

DEFINITIONS.

SECT.
IV.

VII.

Let AWE be a circle in which c is the center, AE a fixed diameter, and D a point in AE or in AE produced; and let the circle AWE roll along the straight line AB so that every part of the circumference may touch it in regular succession from the beginning of the revolution at A to the end of it at B ; the curve $DVFG$ described by the point D during the revolution is called a *Curtate Cycloid* if the point D be without the circle as in Fig. 21. but if D be within the circle as in Fig. 24. the curve $DVFG$ is called a *Prolate or Inflected Cycloid*. Fig. 21.
24

VIII.

The circle AWE is called the *Generating Circle* either of the *Curtate* or *Prolate Cycloid*.

IX.

The straight line DG joining the points in which the generation of the curve begins and ends is called the *Base* of either of the two Cycloids.

X.

The straight line rv bisecting the base DG at right angles and meeting the curve in v is called the *Axis*, and the point v is called the *Vertex* of either of the two cycloids.

XI.

A straight line drawn from any point in the curve perpendicular to the axis is called an *Ordinate* to the axis; and the segment of the axis between the vertex and an ordinate is called an *Absciss*.

PROP. III.

In either the curtate or prolate cycloid the base is equal to the circumference of the generating circle; and in the curtate

SECT.
IV.

curtate cycloid the axis is greater, but in the prolate it is less, than the diameter of the generating circle.

Fig. 21.
24.

For the rest remaining as in the Definitions, the generating circle both at the beginning and end of the revolution touches the straight line AB ; and therefore DAB (18. iii.) is a right angle. And, as at the end of the revolution AD is represented by BG , BG is equal to AD , and ABG is a right angle. Consequently (28. i.) AD , BG are parallel, and therefore (33. i) AB , DG are equal and parallel. But as the revolution of the generating circle begins at A and ends at B , the circumference of the circle is equal to AB , and therefore the base DG is equal to the circumference of the generating circle. Again, as RV bisects DG at right angles, RH is parallel to DA or GB , and therefore AB is bisected in H , and AH is equal to half the circumference of the generating circle. Consequently in the middle of the revolution the point E will coincide with H , and the diameter EA as to position will coincide with the axis RV . Hence it is evident, that in the curtate cycloid the axis RV is greater than the diameter of the generating circle by the double of AD ; but in the prolate cycloid the axis RV is less than the diameter of the generating circle by the double of AD .

Cor. From the above it is evident that in the middle of the revolution the center of the generating circle bisects the axis RV ; and it is also evident that during the revolution the center of the generating circle is at the same distance from the base DG .

PROP. IV.

If an ordinate be drawn from any point in the curtate or prolate

prolate cycloid to the axis and cut a circle described on the axis as a diameter, the arch of the circle between the vertex and ordinate will be to the segment of the ordinate between the cycloid and circle as the circumference of the circle to the base of the cycloid.

SECT.
IV.

Let DVG be a curtate or prolate cycloid, of which DG is the base, rv the axis, and v the vertex, and from any point F in the cycloid let FM be drawn an ordinate to the axis, and let it cut the circle vPR , described upon vr as a diameter, in P ; the arch vp is to the segment FP as the circumference of the circle vPR to the base DG .

Fig. 21.
24.

For, the rest remaining as in the Definitions, let τ be the center of the generating circle when the describing point D coincides with F . Through τ draw the straight line sLa perpendicular to DG , and let it meet AB in L . Then, as AB , DG are parallel, τL is perpendicular to AB ; and, by the Cor. to the preceding Prop. and the description of the curve, τL is equal to the radius of the generating circle, and τa is equal to the radius of the circle vPR . Through the points F and τ draw the straight line $\tau\kappa I$. With τ as a center and τa as a radius let the circle $sFa\kappa$ be described; and with the same center and τL as a radius let the arch LI be described. Then, for the same reasons as in the demonstration of the second Proposition, ML is a parallelogram, FP is equal to HL , and the arch vp is equal to the arch sF . But the arch sF (26. iii.) is equal to the arch $a\kappa$; and as τL is equal to cA , from the generation of either cycloid, the arch LI is equal to the straight line HL . Again, as $\tau a\kappa$, τLI are similar sectors, by Cor. 2. and 3. to the Lemma in the preceding Section, (and II. v.) the arch κa : the arch LI :: circumference of the circle $sFa\kappa$: the

page 268.



L
D

SECT. the circumference of the circle $L I$. Consequently, on
 IV. account of the equals, the arch $v p$: the segment $f p$
 : : the circumference of the circle $v p r$: the base $d g$.

Cor. The equation of the curtate cycloid, and also that of the prolate cycloid, is obtained from the above. For put a = the circumference of the circle $v p r$, b = the base $d g$, z = the arch $v p$, s = $p m$, and y = $f m$; and then $a : b :: z : \frac{bz}{a} = p f$. Consequently $y = \frac{bz}{a} + s$, the equation of the curtate and also that of the prolate cycloid.

SECTION V.

Of the Logarithmic Curve.

DEFINITIONS.

I.

If in the straight line $x y$, of an unlimited length, Fig. 22. segments $A B$, $B C$, $C D$ &c. be taken equal to one another, but indefinitely small, and if from the points of section perpendiculars $A E$, $B F$, $C G$, $D H$ &c. be drawn, and be in geometrical progression; the perpendiculars will be indefinitely near to one another, and the line drawn through their extremities E , F , G , H , &c. is called the *Logarithmic Curve*.

II.

The straight line $x y$ is called the *Axis* to the logarithmic curve, and the perpendiculars $A E$, $B F$, $C G$, $D H$, &c. are called *Ordinates* to it.

PROPOSITION.

The axis is an asymptote to the logarithmic curve.

For, the rest remaining as in the Definitions, let the Fig. 22. ordinates $B F$, $C G$, $D H$, &c. on the right of $A E$ continually increase; and, the segments $A b$, $b c$, $c d$, &c. in the axis being equal to one another, and each equal to $A B$, let the ordinates $b f$, $c g$, $d h$, &c. on the left of $A E$ continually decrease; that is, let $B F$ be to $A E$ as $A E$ to $b f$, and let $A E$, $b f$, $c g$, $d h$, &c. be in geometrical progression. Then ending the first rank, and beginning the second with ordinates equally distant from $A E$, we have the two following ranks of magnitudes proportional taken two and two in the same order.

S

A E

SECT.
V.

$$A E : B F : C G : D H$$

$$d b : c g : b f : A E,$$

and therefore (22. V.) $A E : D H :: d b : A E$; or $D H : A E :: A E : d b$. Hence it is evident that the rectangle under any two ordinates equally distant from $A E$ is equal to the square of $A E$; and therefore if an ordinate on the right of $A E$ be indefinitely great, an ordinate on the left of $A E$, and equally distant from it, will be indefinitely small, but it can never become equal to nothing, or vanish. Consequently the axis $x y$ is an asymptote.

SCHOLIUM.

As $A B, A C, A D, \&c.$ constitute a series in arithmetical progression, and $A E, B F, C G, D H, \&c.$ a corresponding series in geometrical progression, the segments $A B, A C, A D, \&c.$ are analogous to a series of natural numbers, and the ordinates $B F, C G, D H, \&c.$ are analogous to the logarithms of these numbers. The curve is named from these analogies.

The equation of the curve is deduced from the first Definition, in the following manner. Put $A E = 1$, $B F = a$, and then $1 : a :: a : a^2 = C G$; and $a : a^2 :: a^2 : a^3 = D H, \&c.$ Hence it is evident that if x denote any number of equal parts $A B, B C, C D$ in the axis, then will a^x be equal to the ordinate drawn through the extremity of the segment in the axis denoted by x ; and therefore if this ordinate be put equal to y , then $a^x = y$, which is the equation of the curve.

The logarithmic curve, on account of its equation, is also called an *Exponential Curve*.

OF SPIRALS.

SECTION VI.

Of the Logarithmic Spiral.

DEFINITIONS.

I.

If any number of straight lines $CA, CB, CD, CE, \&c.$ Fig. 23. be drawn from the point c within the curve $ABDE$ containing equal angles with it, the curve is called the *Logarithmic Spiral*.

II.

The point c is called the *Center* of the Logarithmic spiral; and any straight line drawn from the center to the curve is called an *Ordinate*.

PROPOSITION.

A number of ordinates of the logarithmic curve are in geometrical progression, if they contain equal angles at the center.

Let $ABDE$ be a logarithmic curve, of which c is Fig. 23. the center and $CA, CB, CD, CE, \&c.$ ordinates, and let the angles $ACB, BCD, DCE, \&c.$ be equal to one another; the ordinates $CA, CB, CD, CE, \&c.$ are in geometrical progression.

or let the angles $ACB, BCD, DCE, \&c.$ be indefinitely small, and then the portions $AB, BD, DE, \&c.$ of the curve may be considered as straight lines, and as,

SECT. VI. by the first Definition, the angles CAB , CBD , CDE , &c. are equal, the triangles ACB , BCD , DCE , &c. are equiangular. Consequently (4. vi.) $AC : BC :: BC : DC$; and $BC : DC :: DC : EC$. The Proposition is therefore evident.

Cor. 1. If the ordinates BC , DC , EC , &c. on the right of AC successively increase, the ordinates $b c$, $d c$, $e c$, &c. on the left of AC will successively decrease. For the rest remaining as above, if the angles ACB , $Ac b$, $b c d$, $d c e$, be equal to one another, it may be proved as above that $BC : AC :: AC : b c$, and $AC : b c :: b c : d c$, &c.

Cor. 2. It may be proved, as in the logarithmic curve, that AC is a mean proportional between ordinates equally distant from it; and therefore as the logarithmic curve cannot meet its asymptote at any assigned distance, so the logarithmic spiral cannot fall into its center at any assigned number of revolutions.

SCHOLIUM.

As the angles ACB , ACD , ACE , &c. constitute a series in arithmetical progression, and the ordinates AC , BC , DC , EC , &c. a series in geometrical progression, the ordinates are analogous to natural numbers, and the angles to their logarithms; and from this consideration the spiral receives its name.

If AC be put $= 1$, $BC = a$, x = any angle in the arithmetical series, and y = the corresponding ordinate, then $a^x = y$, the equation of the logarithmic spiral, for the same reasons as stated in the scholium on the logarithmic curve.

SECTION VII.

Of the Spiral of Archimedes.

DEFINITIONS.

I.

If the straight line cL revolve in a plane about one of its extremities c as a center, with an uniform velocity, and if at the commencement of the revolution a point begin to move from c and proceed in cL towards L with an uniform velocity, the line described by the point moving in cL is called *the Spiral of Archimedes*. Fig. 25.

II.

The line cGA described in the first revolution of cL is called *the First Spiral*; the line AHB described in the second revolution of cL is called *the Second Spiral*; the line BKD described in the third revolution of cL is called *the Third Spiral*, &c.

III.

If A be a fixed point in cL , and cA be the distance moved over by the describing point during the first revolution, the circle described by the point A during the revolution is called *the Generating Circle*; and the point c is called the *Center* of any one of the spirals.

IV.

A straight line drawn from c to the point in which any one of the spirals ends is called *the Axis* of that spiral; and a straight line drawn from c to any other point in the spiral is called *an Ordinate*.

SECT.
VII.

PROP. I.

If an ordinate be drawn to any point in the first spiral, and be produced till it cut the generating circle, the ordinate will be a fourth proportional to the circumference of the generating circle, the axis, and the circular arch generated from the beginning of the revolution to the point of section.

Fig. 26.

Let AG be an ordinate to the first spiral AGB , of which $BKHZ$ is the generating circle, and A being the center, let AB be the axis, and let AG be produced and cut the circle in H ; the ordinate AG is a fourth proportional to the circumference $BKHZ$, the axis AB , and the circular arch BKH , generated from the beginning of the revolution to H the point of section.

For the line in which the fixed point B is situated revolves about A with an uniform velocity, and the point describing the spiral moves from A towards B with an uniform velocity, according to the third and first Definitions. The circumference of the circle therefore and the axis AB are described by uniform velocities in the same time; and for the same reasons the arch BKH and the ordinate AG are described by the same uniform velocities in the same time. But spaces described by the same uniform velocity are to one another as the times in which they are described, and therefore the circumference of the circle $BKHZ$ is to the arch BKH as the time of one revolution to the time of describing the arch BKH , and AB is to AG as the same portions of time are to one another. Consequently (II. v. and alternation) the circumference of the circle $BKHZ$ is to the axis AB as the arch BKH to the ordinate AG .

Cor. I. The rest remaining, if AF another ordinate be produced and cut the generating circle in Z , by the
above

Cor. 2. If the axis of the first spiral, or, which is the same thing, the radius of the generating circle, be put equal to r , the circumference of the generating circle equal to c , any arch BKH from the axis equal to z , and the corresponding ordinate AG equal to y , then $c : r :: z : \frac{rz}{c} = y$, the equation of the first spiral.

PROP. II.

An ordinate drawn to any point in the second spiral is a fourth proportional to the circumference of the generating circle, the radius of the circle, and the sum of the circumference of the circle and its arch generated from the beginning of the revolution to the ordinate.

Let AE be an ordinate drawn to any point E in the second spiral $BEDM$, and let it cut the generating circle $BKHZ$ in H . Let A be the center and AM the axis, and consequently the situation from which the revolution begins. The circumference $BKHZ$ of the generating circle is to its radius AB as the sum of the circumference $BKHZ$ and the arch BKH to the ordinate AE . Fig. 26.

For, as AL revolves with an uniform velocity, the circumference $BKHZ$ is to the sum of the circumference $BKHZ$ and the arch BKH as the time of one revolution to the time in which the circumference and the arch BKH are generated by B the extremity of the radius. And these portions of time are to one another as the radius AB to the ordinate AE , as the point moving in AL with an uniform velocity passes over AB , AE in these portions of time respectively. Consequently (II. v. and alternation) the circumference $BKHZ$:
the

SECT. VII. the radius AB : : the circumference $BKHZ$ + the arch BKH : the ordinate AE .

Cor. 1. The rest remaining as above, if AD another ordinate to the spiral $BEDM$ cut the generating circle in Z , then, by the above, (and 11. v.) the circumference $BKHZ$ + the arch BKH : AE : : the circumference $BKHZ$ + the arch BKZ : AD .

Cor. 2. Put c = the circumference of the generating circle, r = its radius, y = the ordinate AE , and z = the arch BKH ; and then, by the Proposition, $c + z$: y : : c : r , and therefore the equation of the second spiral is $r \times c + z = cy$.

Cor. 3. If c denote the circumference of the generating circle, r its radius, y an ordinate drawn to any point in the n th spiral, and z the arch of the generating circle generated from the beginning of the revolution to the ordinate, it may be proved, as above, that $c : r$: : $\overline{n-1} \times c + z$: y . Consequently $r \times \overline{nc - c + z} = cy$ is a general equation for any one of the spirals of Archimedes.

Cor. 4. The rest of the notation remaining as above, let an ordinate v contain an angle with the ordinate y , and let d denote the arch in the generating circle which measures this angle. Let z denote an arch of the generating circle, generated from the beginning of the revolution to the ordinate r ; and let the ordinate y and the ordinate x contain an angle equal to the angle contained by y and v , and consequently also measured by an arch d in the generating circle. Then by the last Cor. (and 11. v.) $\overline{n-1} \times c + z$: y : : $\overline{n-1} \times c + z + d$: v : : $\overline{n-1} \times c + z$: r : : $\overline{n-1} \times c + z + d$: x ; and therefore, by Lemma X. page 154. (and 11. v.) $d : v - y$: : $d : x - r$. Consequently (14. v.) the

the excess of v above y is equal to the excess of x above y . SECT.
VII.

Cor. 5. From the last Cor. it is evident, that if any number of ordinates contain angles which constitute a series in arithmetical progression, the ordinates themselves will be in arithmetical progression.

SECTION VIII.

Of the Hyperbolic or Reciprocal Spiral.

DEFINITIONS.

I.

Fig. 27. If with c , one of the extremities of the straight line cL , as a center and distances cA , cB , cD , &c. in cL , arches of circles AG , BH , DI , &c. be described, and if these arches be equal to one another, and indefinitely near; the curve GHI , &c. drawn through their extremities is called *the Hyperbolic or Reciprocal Spiral*.

II.

The straight line cL is called the *Axis* of the hyperbolic spiral, the point c its *Center*, and any straight line ck drawn from the center to the curve is called an *Ordinate*.

PROP. I.

In the hyperbolic spiral any two ordinates are reciprocally to one another as the angles they contain with the axis.

Fig. 28. Let EGI be a hyperbolic spiral, of which c is the center, cL the axis, and cI , cG any two ordinates; cI is to cG as the angle LcG to the angle LcI .

For let the circular arches GA , ID be described, and let them meet the axis in A and D . Let cG produced meet the arch DI produced in M ; and then as DCM , ACG are similar sectors, by Cor. 3. to the Lemma in the third section, $CD : cA ::$ the arch DM : the arch AG . But cI is equal to cD , and cG to cA ,
and

and by the first Definition the arch AG is equal to the arch DI ; and therefore $CI : CG ::$ the arch DM : the arch DI . Again (33. vi.) the arch DM : the arch $DI ::$ the angle DCM or LCG : the angle DCI or LCI . Consequently (11. v.) CI is to CG as the angle LCG to the angle LCI .

SECT.
VIII.

Cor. Every thing remaining as above, let CI be a given ordinate, and put it equal to r ; put the arch $DI = a$, $CG = y$, and the arch $DM = z$. Then, by the above, $r : y :: z : a$, and $ar = yz$, the equation of the curve.

PROP. II.

If from the center of an hyperbolic spiral a straight line be drawn at right angles to the axis and equal to one of the circular arches intercepted between the curve and axis, a straight line drawn through the extremity of this perpendicular parallel to the axis will be an asymptote to the curve.

— Let EGI be an hyperbolic spiral, of which c is the center and CL the axis, and let the straight line CB at right angles to CL be equal to ID any circular arch intercepted between the curve and the axis; the straight line BF drawn through B parallel to CL is an asymptote to the curve. Fig. 28.

For through I draw NF parallel to CB , or perpendicular to CL , and let it meet BF in F ; and then NF (34. i.) is equal to CB , and therefore equal to the arch DI . But the arch DI is greater than its sine NI , and therefore the point F in BF is without the curve. Again, from the preceding Proposition it is manifest, that if the ordinate CI be indefinitely great when compared to another ordinate CG , the angle DCI is indefinitely small when compared to ACG ; and when an angle

SECT. angle is indefinitely small, the excess of the arch which
VIII. measures it above its sine is less than any assigned magnitude. Consequently as IF is the excess of the arch DI above its sine NI , if the curve EGI be continued till the angle DCI become indefinitely small, the distance IF of the curve from BF will be less than any assigned magnitude. The straight line BF is therefore an asymptote to the curve.

THE END.

